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INTRODUCTION TO COALGEBRA

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Abstract. A survey of parts of General Coalgebra is presented with applications to the theory of systems. Stress is laid on terminal coalgebras and coinduction as well as iterative algebras and iterative theories.

1. Preface: Is this Really an Introduction to Coalgebra?

For my series of lectures on coalgebra on the preconference to CTCS 2002 I prepared a short text called "Introduction to Coalgebra" that was intended to cover the material of the series. I was not worried about the the choice of topics: they were given by the intentions of the course. Two years later the Program Chair Rick Blute suggested that I could publish the text in the proceedings—and after I agreed, I found myself in a fix: what topics to choose? If I add to the existing text the minimum of material that would indeed constitute a general introduction to coalgebra, the text would grow immensely. Besides, excellent introductions exist already, see e.g. [R1], [G], [JR]. I decided for a minimalistic approach: in the following sections I try and present an introduction to the parts of coalgebra which are those "dearest to my heart", and to which I have also contributed. Thus, a substantial part of coalgebra topics is not mentioned at all; I hope the reader will enjoy what I present below, and I ask her or him not to take a missing topic as a message of any kind. Almost all proofs are omitted, with precise references provided, just some simple, instructive proofs are left.

Contents

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2. Introduction: Systems as Coalgebras

The present section, closely following ideas of J. Rutten [R1], tries and explains the motivation of General Coalgebra which stems from various types of dynamical systems.

2.1 The earliest observation in the literature about application of coalgebras is probably due to M. Arbib and E. G. Manes [AMa] concerning deterministic automata described by a set Q (of states), a next-state function $\delta\colon Q \times \Sigma \longrightarrow Q$ (where Σ is the set of inputs) and a predicate *final*: $Q \longrightarrow$ **bool**. Initial states are ignored at the moment, see later. We can use the curried version of the next-state function, say $\bar{\delta}\colon Q \longrightarrow Q^{\Sigma}$ where $\bar{\delta}(q)$ is the function $\delta(q,-)\colon \Sigma \longrightarrow Q$ for every state $q \in Q$. Then $\bar{\delta}$ and *final* yield together one function from Q to $Q^{\Sigma} \times$ **bool**. In other words, a deterministic automaton consists of a set Q and a "dynamics"

$$\alpha\colon Q \longrightarrow HQ \qquad \text{where} \qquad HQ = Q^{\Sigma} \times \mathbf{bool}\,.$$

Observe that there is a canonical way of making a functor $H\colon$ **Set** $\longrightarrow$ **Set** out of the above rule $HQ = Q^{\Sigma} \times$ **bool** for objects Q: given a morphism $h\colon Q_1 \longrightarrow Q_2$ then $Hh\colon Q_1^{\Sigma} \times \mathbf{bool} \longrightarrow Q_2^{\Sigma} \times \mathbf{bool}$ is the function which to every pair (u,x) with $u\colon \Sigma \longrightarrow Q_1$ and $x \in$ **bool** assigns the pair $(h{\cdot}u, x)$.

2.2 Deterministic automata constitute an example of coalgebras specified by an endofunctor H of **Set**: a *coalgebra* is a pair (Q,α) consisting of a set Q (of "states") and a function $\alpha\colon Q \longrightarrow HQ$ ("dynamics"). Given coalgebras (Q,α) and (Q',α'), a *coalgebra homomorphism* is a function $f\colon Q \longrightarrow Q'$ such that the square

(1)
$$\begin{array}{ccc} Q & \xrightarrow{\ \alpha\ } & HQ \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle Hf} \\ Q' & \xrightarrow[\ \alpha'\ ]{} & HQ' \end{array}$$

commutes. For example in the above case of $H = (-)^{\Sigma} \times$ **bool**, this is precisely the concept of functional simulation of deterministic automata.

2.3 An important example of coalgebras are *labeled transition systems* consisting of a state set Q and transitions

$$q \xrightarrow{\ s\ } \bar{q} \qquad \text{for } q, \bar{q} \in Q \text{ and } s \in \Sigma$$

(where Σ is the set of possible actions). More precisely, for every action s a binary relation $\xrightarrow{\ s\ }$ is given on Q. This can be viewed as a coalgebra for the functor

$$H = \mathscr{P}(\Sigma \times -)$$

where $\mathscr{P}\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ is the *power-set functor*. (Recall that for every set Q, the power-set $\mathscr{P}Q$ consists of all subsets of Q, and for every function $f\colon Q_1 \longrightarrow Q_2$ the function $\mathscr{P}f$ maps $M \subseteq Q_1$ to $f[M] \subseteq Q_2$.) In fact, define

$$\alpha\colon Q \longrightarrow \mathscr{P}(\Sigma \times Q)$$

by assigning to every state q the set $\alpha(q)$ of all pairs $(s, \bar{q}) \in \Sigma \times Q$ with $q \xrightarrow{s} \bar{q}$. Coalgebra homomorphisms $f\colon (Q, \alpha) \longrightarrow (Q', \alpha')$ are precisely the functions which preserve and reflect transitions. That is

(a) $q \xrightarrow{s} \bar{q}$ in Q implies $f(q) \xrightarrow{s} f(\bar{q})$ in Q'

and

(b) $f(q) \xrightarrow{s} \bar{q}$ in Q' implies $q \xrightarrow{s} \hat{q}$ in Q for some $\hat{q} \in Q$ with $f(\hat{q}) = \bar{q}$.

These homomorphisms serve e.g. for a simple formulation of a strong bisimulation, see Section 5.

2.4 Just as in algebraic semantics the initial algebra plays a central role, in coalgebra the *terminal coalgebras* (i.e., terminal objects of the category of all coalgebras) are of major importance. Recall that a terminal coalgebra is a coalgebra $T \xrightarrow{\tau} HT$ such that for every coalgebra $Q \xrightarrow{\alpha} HQ$ there exists a unique homomorphism $[\![\alpha]\!]\colon Q \longrightarrow T$. Usually, T is the set of all "types of behaviors" of states and $[\![\alpha]\!]$ assigns to $q \in Q$ the type of behavior q has.

2.5. Example. $H = (-)^\Sigma \times \mathbf{bool}$ (deterministic automata). Here $T = \mathscr{P}\Sigma^*$ is the set of all formal languages, considered as an automaton as follows: for a language $L \in \mathscr{P}\Sigma^*$ and $s \in \Sigma$ put

$$\delta(L, s) = \{w \in \Sigma^*; sw \in L\};$$

and $\mathit{final}(L) = \text{true}$ iff L contains ε.

Given an automaton (Q, α) the unique homomorphism

$$[\![\alpha]\!]\colon Q \longrightarrow T = \exp \Sigma^*$$

assigns to every state q the language $[\![\alpha]\!](q) \subseteq \Sigma^*$ which the automaton Q accepts in case q is the initial state. Here we see the manner in which the (so far missing) concept of initial state is recaptured.

2.6. Example. $HX = X \times X + 1$. This is an endofunctor whose coalgebras are deterministic systems with a binary input and deadlock states (i.e., states that do not react to input). In fact, each such system is described by the dynamics

$$\alpha\colon Q \longrightarrow Q \times Q + 1$$

taking deadlock states to the right-hand summand, and non-deadlock states q to the pair $\alpha(q) = (q_0, q_1)$ of next states. Homomorphisms are the functional bisimulations, i.e., maps preserving and reflecting input reaction and preserving and reflecting deadlock states.

A terminal coalgebra T consists of all (finite and infinite) binary trees[1]: the unique deadlock of T the trivial single-node tree, the next states of a nontrivial tree are the two maximal proper subtrees.

Given a system (Q, α) the unique homomorphism

$$[\![\alpha]\!]\colon Q \longrightarrow T$$

takes every state q to the tree $[\![\alpha]\!](q) \in T$ obtained by the unfolding of q in the system Q.

2.7. Example. $H = \mathscr{P}_f$, the finite-power-set functor. This functor assigns to every set X the set $\mathscr{P}_f X$ of all finite subsets of X, and behaves on morphisms as the power-set functor $\mathscr{P}$ does.

A coalgebra of $\mathscr{P}_f$ can be viewed as a finitely branching nondeterministic system: the dynamics

$$\alpha\colon Q \longrightarrow \mathscr{P}_f Q$$

assigns to every state q the collection $\alpha(q)$ of all possible next states. Sometimes one also identifies Q with a finitely branching directed graph: $\alpha(q)$ is the set of all neighbour nodes of q. However, this is often not a reasonable point of view because the coalgebra homomorphisms are much stronger than graph homomorphisms: given two systems (Q, α) and (Q', α'), a coalgebra homomorphism is a function $h\colon Q \longrightarrow Q'$ which preserves and reflects the dynamics. That is, h is a graph homomorphism such that if $\bar{q}$ is a next state of $h(q)$ in Q', then there exists a next state $\hat{q}$ of q in Q with $\bar{q} = h(\hat{q})$.

A final coalgebra of $\mathscr{P}_f$ can be described as the coalgebra of all strongly extensional, finitely branching trees, see Example 5.10(iv).

2.8. Example. $H = \mathscr{P}(\Sigma \times -)$. As we will see in Section 3, this endofunctor of **Set** does not have a terminal coalgebra. However, *every* endofunctor H of **Set** has a terminal coalgebra in **Class**, the category of classes, see Section 10. For example, the terminal coalgebra of $\mathscr{P}(\Sigma \times -)$ is the coalgebra T of all nonordered trees with nodes labeled in Σ modulo tree bisimilarity.

Given a labeled transition system (Q, α), the unique homomorphism

$$[\![\alpha]\!]\colon Q \longrightarrow T$$

assigns to every state q the bisimilarity class of the unfolding tree of q in Q.

[1]Trees are understood to be rooted and they are considered up-to isomorphism. Unless explicitly stated we consider ordered trees, i.e., with a linear order on the set of children of every node.

2.9 Terminal coalgebras of "derived" endofunctors are also of some importance: for every set Y a terminal coalgebra, CY, of the endofunctor $H(-)\times Y$ is a cofree coalgebra colored in Y, a concept explained in Section 3. A terminal coalgebra, TY, of the endofunctor $H(-)+Y$ is a free completely iterative algebra on Y—the concept of (completely) iterative algebras and their relationship to coalgebra is explained in Sections 6–9. The last section is devoted to coalgebra in the category of classes: this category has a number of convenient properties, e.g., every endofunctor H has a terminal coalgebra (and, consequently, the above coalgebras CY and TY always exist).

2.10 Although all important examples of application of coalgebra seem to concern coalgebras in **Set**, there are good reasons to develop the whole theory in an abstract category, e.g., (1) the dualization, see 3.6, only works in the abstract setting, (2) the techniques used are often clearer in the abstract categorical formulation, and (3) the category of classes may actually be preferable to **Set**.

3. Categories of Algebras and Coalgebras

3.1. Algebras. Let $\mathscr{A}$ be a category. For every endofunctor $H\colon \mathscr{A} \longrightarrow \mathscr{A}$ by an *H-algebra* is meant a pair consisting of an object $A \in \mathscr{A}$ and a morphism $\alpha\colon HA \longrightarrow A$. Given H-algebras $HA \xrightarrow{\alpha} A$ and $HB \xrightarrow{\beta} B$, by a *homomorphism* is meant a morphism $h\colon A \longrightarrow B$ of $\mathscr{A}$ for which the following square

$$\begin{array}{ccc} HA & \xrightarrow{\alpha} & A \\ {\scriptstyle Hh}\downarrow & & \downarrow{\scriptstyle h} \\ HB & \xrightarrow[\beta]{} & B \end{array}$$

commutes.

Example: for the set functor $HX = (X \times X) + 1$ an H-algebra is a set A together with one binary operation, $A \times A \longrightarrow A$, and one constant, $1 \longrightarrow A$, represented by the corresponding morphism $\alpha\colon A \times A + 1 \longrightarrow A$. Homomorphisms are functions preserving the two operations in the usual algebraic sense.

We denote by $\mathbf{Alg}\, H$ the category of all H-algebras and homomorphisms. They were for the first time considered by J. Lambek [L] who proved the following

3.2. Lambek Lemma. *(An initial H-algebra is a fixed point of H.) If $HI \xrightarrow{\varphi} I$ is an initial object of $\mathbf{Alg}\, H$, then φ is an isomorphism.*

Proof. For the H-algebra $H(HI) \xrightarrow{H\varphi} HI$ there exists a unique homomorphism

$$\begin{array}{ccc} HI & \xrightarrow{\varphi} & I \\ {\scriptstyle Hh}\downarrow & & \downarrow{\scriptstyle h} \\ H(HI) & \xrightarrow{H\varphi} & HI \\ {\scriptstyle H\varphi}\,\vdots & & \vdots\,{\scriptstyle \varphi} \\ HI & \xrightarrow[\varphi]{} & I \end{array}$$

Then φh is an endomorphism of algebra I—thus $\varphi h = \mathrm{id}$, since I is initial. The upper square of the above diagram yields $h\varphi = F(\varphi h) = \mathrm{id}$. ∎

3.3. Examples. (i) The power-set functor $\mathscr{P}$ does not have an initial algebra: by the Cantor Theorem $\mathscr{P}$ has no fixed point (since $\operatorname{card} \mathscr{P}I > \operatorname{card} I$).

(ii) The functor $HX = (X \times X) + 1$ has an initial algebra I consisting of all finite binary trees. The binary operation is that of tree-tupling, and the constant is the single-node tree.

(iii) For the set-functor $HX = \Sigma \times X + 1$ an H-algebra is a unary algebra with operation symbols from Σ and one constant. An initial algebra is the set Σ^* of all finite words (= sequences) over Σ, with the operations $\Sigma \times \Sigma^* \longrightarrow \Sigma^*$ of concatenation, $(\sigma_1 \dots \sigma, \sigma_n) \longmapsto (\sigma\sigma_1 \dots \sigma_n)$ and the constant ε.

3.4. Coalgebras. A pair (A, α) consisting of an object A and a morphism $\alpha\colon A \longrightarrow HA$ is called an H-coalgebra. We call α the *dynamics* of the coalgebra A. Given H-coalgebras (A, α) and (B, β), by a *homomorphism* is meant a morphism $h\colon A \longrightarrow B$ in $\mathscr{A}$ for which the following square

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & HA \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Hh} \\ B & \xrightarrow[\beta]{} & HB \end{array}$$

commutes.

3.5. Notation. We denote by $\mathbf{Coalg}\, H$ the category of all coalgebras and homomorphisms.

In case a *terminal coalgebra* $T \xrightarrow{\tau} HT$, i.e., a terminal object of $\mathbf{Coalg}\, H$, exists, we denote for every coalgebra $\alpha\colon A \longrightarrow HA$ by $[\![\alpha]\!]\colon A \longrightarrow T$ the unique homomorphism

$$\begin{array}{ccc} T & \xrightarrow{\ \tau\ } & HT \\ {\scriptstyle [\![\alpha]\!]}\big\downarrow & & \big\downarrow{\scriptstyle H[\![\alpha]\!]} \\ A & \xrightarrow[\ \alpha\ ]{} & HA \end{array}$$

3.6. Observation. If we form the obvious functor $H^{\mathrm{op}}\colon \mathscr{A}^{\mathrm{op}} \longrightarrow \mathscr{A}^{\mathrm{op}}$, then the categories $\mathbf{Alg}\, H^{\mathrm{op}}$ and $\mathbf{Coalg}\, H$ are dual to each other.

3.7. Corollary. *Terminal H-coalgebras are fixed points of H.*

In fact, this follows from Lambek Lemma by duality.

3.8. Examples. Coalgebras over endofunctors of **Set**.

(i) For $HX = X + 1$ a coalgebra $A \xrightarrow{\alpha} A + 1$ is precisely a set A with a partial unary operations. Homomorphisms $h\colon A \longrightarrow B$ of coalgebras are the "strong" homomorphisms of partial unary algebras, i.e., given $a \in A$ then the A-operation is defined in A iff the B-operation is defined in $h(a)$.

A final coalgebra can be described as the coalgebra

$$\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$$

of extended natural numbers whose dynamics is the following predecessor function

$$\mathrm{pred}(0) \quad \text{undefined}$$

$$\mathrm{pred}(n) = n - 1 \qquad \text{for } 0 < n < \infty$$

and

$$\mathrm{pred}(\infty) = \infty.$$

Given a coalgebra A, the unique homomorphism $[\![\alpha]\!]\colon A \longrightarrow \overline{\mathbb{N}}$ assigns to every state a the maximum n such that all $\alpha^i(a)$ for $1 \le i < n$ are defined.

(ii) For $HX = \Sigma \times X$ a coalgebra $A \xrightarrow{\alpha} \Sigma \times A$ consists of two functions $A \longrightarrow \Sigma$ and $A \longrightarrow A$. A terminal coalgebra is the set

$$T = \Sigma^\omega$$

of all infinite sequences in Σ with the dynamics given by the functions "head" and "tail":

$$\tau\colon T \longrightarrow \Sigma \times T, \qquad \tau(x_0, x_1, x_2, \dots) = \big(x_0, (x_1, x_2, x_3, \dots)\big).$$

(iii) For $HX = X \times X + 1$ a coalgebra $A \xrightarrow{\alpha} A \times A + 1$ is given by a partial function from A to $A \times A$. A terminal coalgebra has been described the Example 2.6: T consists of all binary trees.

(iv) $\mathscr{P}$ has no terminal coalgebras.

3.9. Remark. The examples (i)–(iii) in 3.8 are special cases of the following situation: suppose a *signature* Σ, i.e., a set of operation symbols σ with prescribed arities $\operatorname{ar}\sigma \in \mathbb{N}$, is given. We denote by $H_\Sigma\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ the following *polynomial functor*

$$H_\Sigma X = \Sigma_0 + (\Sigma_1 \times X) + (\Sigma_2 \times X \times X) + \cdots$$

where $\Sigma_n \subseteq \Sigma$ is the set of all n-ary symbols. Elements of $H_\Sigma X$ are denoted by $\sigma(x_1, \dots, x_n)$ for $\sigma \in \Sigma_n$.

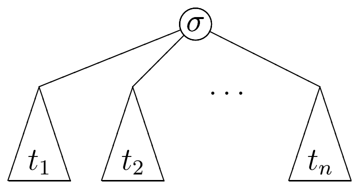
H_Σ-algebras are the usual Σ-algebras of General Algebra, i.e., sets A equipped with an n-ary operation $\sigma_A\colon A^n \longrightarrow A$ for every $\sigma \in \Sigma_n$ (or, equivalently, with a function from $H_\Sigma A$ to A).

H_Σ-coalgebras are deterministic automata with a state set Q, an output set Σ and with n next states for every state with an n-ary output.

An initial algebra, I_Σ, can be described as the algebra of all finite Σ-*trees*, i.e., ordered trees with all nodes labeled in Σ so that a label in Σ_n implies that the node has precisely n children. The operation

$$H_\Sigma(I_\Sigma) \longrightarrow I_\Sigma$$

is *tree-tupling*: given $\sigma \in \Sigma_n$ and Σ-labeled trees $t_1, \dots, t_n$, form the Σ-labeled tree



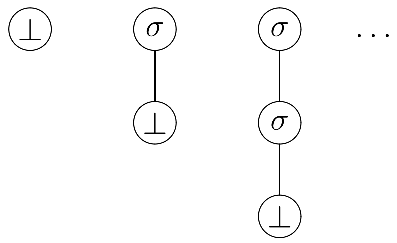
We now denote by

$$T_\Sigma$$

the analogous coalgebra obtained by dropping the finiteness requirement: T_Σ has as elements all (finite or infinite) Σ-labeled trees. We again have a structure of an algebra $H_\Sigma(T_\Sigma) \longrightarrow T_\Sigma$ given by tree tupling. This (as in I_Σ above) is an isomorphism, so we can invert it:

3.10. Theorem. (See [AP1].) *The coalgebra T_Σ of all Σ-labeled trees whose dynamics is "parsing", i.e., the inverse of tree-tupling, is terminal for H_Σ.*

3.11. Example. The functor $HX = X + 1$ of 3.7(i) is polynomial with $\Sigma_0 = \{\bot\}$, $\Sigma_1 = \{\sigma\}$ and $\Sigma_n = \emptyset$ for $n > 1$. The coalgebra T_Σ consists of the following trees



and the unique infinite tree with all nodes labeled by σ. This is isomorphic to $\overline{\mathbb{N}}$ as described in 3.7(i).

3.12. Remark. (a) Recall that the concept of a *free algebra* on an object X (of generators): it is an algebra

$$\varphi_X \colon HX^{\#} \longrightarrow X^{\#}$$

together with a universal arrow

$$\eta_X \colon X \longrightarrow X^{\#}$$

of the canonical forgetful functor $\mathbf{Alg}\, H \longrightarrow \mathscr{A}$. This means that for every algebra A and every morphism $f \colon X \longrightarrow A$ (interpretation of generators) there exists a unique homomorphism

$$f' \colon X^{\#} \longrightarrow A \qquad \text{with} \qquad f = f' \cdot \eta_X.$$

(b) H is called a *varietor*, see [AT1], provided that free algebras exist, i.e., the above forgetful functor of $\mathbf{Alg}\, H$ has a left adjoint $X \longmapsto X^{\#}$.

For example, a nonconstant functor $H \colon \mathbf{Set} \longrightarrow \mathbf{Set}$ is a varietor iff H has arbitrarily large fixed points, see [AT1].

(c) It is easy to verify that the Lambek Lemma implies that $X^{\#}$ is a coproduct

$$X^{\#} = HX^{\#} + X \qquad \text{with injections } \varphi_X \text{ and } \eta_X.$$

In fact, $X^{\#}$ is an initial algebra of the functor $H(-) + X$ (and vice versa: an initial algebra of $H(-) + X$ is a free H-algebra on X).

(d) If H is a varietor, then the monad of free H-algebras was elegantly characterized by M. Barr [B1] as the free monad on H. Also conversely: if an endofunctor of a complete, well-powered category generates a free monad, it must be a varietor. Moreover, $\mathbf{Alg}\, H$ is monadic over $\mathscr{A}$, being equivalent to the Eilenberg-Moore category of the free-algebra monad.

3.13. Remark. Dually H is a *covarietor* if cofree coalgebras exist (or, if H generates a cofree comonad).

Explicitly, for every object X (of "colors") in $\mathscr{A}$ a *cofree coalgebra* on colors from X is a coalgebra

$$\psi_X \colon X_{\#} \longrightarrow HX_{\#}$$

together with a universal "coloring" morphism

$$\varepsilon_X \colon X_{\#} \longrightarrow X;$$

for every coalgebra (A, α) and every coloring morphism $f \colon A \longrightarrow X$ there exists a unique homomorphism

$$f' \colon A \longrightarrow X_{\#} \qquad \text{with} \qquad f = \varepsilon_X \cdot f'.$$

The interpretation is that we can use colors (elements of X) to color the states of systems. We cannot observe individual states, but can observe their colors. Then $X_{\#}$ is the set of all types of behaviors of states of systems colored in X.

3.14. Examples. (1) For $HX = X \times X + 1$ a cofree coalgebra $X_\#$ is the coalgebra of all binary trees with nodes colored in X. Given a system (Q, α) and a coloring $f\colon Q \longrightarrow X$, the unique homomorphism $f'\colon Q \longrightarrow X_\#$ assigns to every state q the colored tree obtained by unfolding q in the system Q and observing the colors of the future states.

(2) More generally, for every polynomial functor H_Σ a cofree coalgebra $X_\#$ is the coalgebra of all Σ-trees with an additional coloring of the nodes in X. This is, obviously, just the terminal coalgebra of the polynomial functor $H_{\Sigma[X]}$ where the new signature $\Sigma[X]$ has operation symbols from $\Sigma \times X$ and the arity of every pair (σ, x) is n for $\sigma \in \Sigma_n$.

(3) A cofree H-coalgebra on X is precisely a terminal coalgebra of $H(-) \times X$. This statement, dual to 3.11(c), is illustrated by the above example of $H_{\Sigma[X]} = H_\Sigma(-) \times \Sigma$.

(4) If 1 is a terminal object of $\mathscr{A}$ then

$$1_\# = T$$

is a terminal coalgebra.

(5) The power-set functor $\mathscr{P}$ does not have cofree coalgebras. But it has them in the category of classes, see Section 10.

3.15. Remark. (a) Recall that a functor is called *finitary* if it preserves filtered colimits, and *accessible* if it preserves, for some infinite cardinal λ, λ-filtered colimits. In case $\mathscr{A} = \mathbf{Set}$, accessible endofunctors are precisely the *bounded* endofunctors of [KM] (as proved in [AP2]). That is, such that for some cardinal λ every coalgebra is a union of subalgebras of cardinalities at most λ.

(b) Every accessible endofunctor of **Set** is a varietor and a covarietor; however, there are varietors and covarietors which are not accessible, see [AT1].

(c) Recall that an object A of a category $\mathscr{A}$ is called *finitely presentable* provided that its hom-functor $\hom(A, -)\colon \mathscr{A} \longrightarrow \mathbf{Set}$ is finitary. The category $\mathscr{A}$ is called *locally finitely presentable* in the sense of Gabriel and Ulmer [GU], see also [AR], provided that

1. $\mathscr{A}$ has colimits

and

2. $\mathscr{A}$ has a small set of finitely presentable objects whose closure under filtered colimits is all of $\mathscr{A}$.

For example in **Set** "finitely presentable" means finite, and every set is a directed colimit of finite sets. Thus, **Set** is locally finitely presentable. In an equationally definable class of algebras "finitely presentable" has the classical meaning (of presentation by finitely many generators and finitely many equations), and each such category is locally finitely presentable. In contrast, the category of domains and continuous functions is not locally finitely presentable: except trivial domains no object is finitely presentable.

(d) Generalizing (b), every accessible endofunctor of a locally finitely presentable category is a varietor and a covarietor, see [B2].

3.16. Example. The polynomial functor H_Σ is a (finitary) varietor and a covarietor. A free algebra $X^{\#}$ is the algebra of all finite Σ-*trees on* X (where "on X" means that leaves are labeled by nullary operation symbols or by elements of X). A cofree coalgebra $X_{\#}$ is the coalgebra of all finite and infinite Σ-trees with an additional coloring of all nodes in X.

3.17. Theorem. (See [A0].) *Let $\mathscr{A}$ have and H preserve ω-colimits. If 0 is the initial object of $\mathscr{A}$ and $!\colon 0 \longrightarrow H0$ the unique morphism, then an initial algebra I is a colimit of the ω-chain*

$$0 \xrightarrow{\ !\ } H0 \xrightarrow{\ H!\ } HH0 \xrightarrow{\ HH!\ } \cdots$$

3.18. Corollary. *If $\mathscr{A}$ has and H preserves ω^{op}-limits then a terminal coalgebra T is a limit of*

$$1 \xleftarrow{\ !\ } H1 \xleftarrow{\ H!\ } HH1 \xleftarrow{\ HH!\ } \cdots \qquad (1 \textit{ terminal in } \mathscr{A}).$$

3.19. Theorem. (Initial-Algebra Construction, see [A0].) *Let $\mathscr{A}$ have and H preserve λ-colimits (λ an infinite cardinal). Then an initial algebra is $I = \mathrm{colim}_{i<\lambda} W_i$ for the λ-chain determined (essentially uniquely) as follows:*

Initial step: $W_0 = 0$, $W_1 = H0$ and $W_{01} = !\colon 0 \longrightarrow H0$.

Isolated step: $W_{i+1} = HW_i$ and $W_{i+1,j+1} = HW_{i,j}\colon HW_i \longrightarrow HW_j$.

Limit step: $W_j = \mathrm{colim}_{i<j} W_i$ for limit ordinals j.

3.20. Corollary. (Terminal-Coalgebra Construction.) *If $\mathscr{A}$ has and H preserves λ^{op}-limits, then a terminal algebra is $T = \lim_{i<\lambda} W_i$ for the dual λ^{op}-chain ($W_0 = 1$, $W_{i+1} = HW_i$ and $W_j = \lim_{i<j} W_i$ for limit ordinals j).*

3.21. Theorem. (See [W1].) *A finitary functor $H\colon$ **Set** $\longrightarrow$ **Set** requires only $\omega + \omega$ steps of the terminal-coalgebra construction:*

$$T = \lim_{i<\omega+\omega} W_i.$$

Example. (See [W1].) The terminal coalgebra-construction of $\mathscr{P}_f$ yields after ω steps all, not necessarily finitely branching, strongly extensional trees as W_ω, compare 5.10(iv). Then $W_{\omega+1,\omega}\colon \mathscr{P}_f W_\omega \hookrightarrow W_\omega$ is the subset of all trees finitely branching at level 1, $W_{\omega+2,\omega}\colon \mathscr{P}_f W_{\omega+1} \hookrightarrow W_\omega$ are all trees finitely branching at levels 1 and 2, etc. Thus $T = \lim_{i<\omega+\omega} W_i$ is the algebra of all finitely branching strongly extensional trees.

3.22. Remark. (i) Whenever H is an accessible endofunctor, then so is $H(-)+Y$ for every object Y. Consequently, a terminal coalgebra TY of the latter functor exists.

By Lambek's Lemma, the dynamics $TY \longrightarrow HTY + Y$ is an isomorphism. In other words, TY is a coproduct of HTY and Y; we denote by

$$\tau_Y\colon HTY \longrightarrow TY \qquad (TY \text{ is an } H\text{-algebra})$$

and

$$\eta_Y : Y \longrightarrow TY \qquad (TY \text{ contains } Y)$$

the coproduct injections.

In general, we use inr and inl for coproduct injections.

(ii) Recall that if $Y^\#$ is a free H-algebra on Y, then for every morphism $s\colon X \longrightarrow Y^\#$ there exists a unique homomorphism $\hat{s}\colon X^\# \longrightarrow Y^\#$ extending s. We prove now that TY has a similar property; we include a full proof because it is simple and instructive. The result also follows from Theorem 8.4 below.

3.23. Substitution Theorem. (See [Mo] or [AAMV].) *For every accessible functor H and every morphism $s\colon X \longrightarrow TY$ there exists a unique homomorphism $\hat{s}\colon TX \longrightarrow TY$ of H-algebras extending s. (That is, a unique $\hat{s}$ with $s = \hat{s}\cdot\eta_x$ and $\hat{s}\cdot\tau_X = \tau_Y\cdot H\hat{s}$.)*

Proof. Consider the following dynamics α on $TX + TY$ for the endofunctor $H(-) + Y$: α is a composite of

$$TX + TY = HTX + X + TY \xrightarrow{\text{id} + [s,\text{id}]} HTX + TY = HTX + HTY + Y$$

followed by the canonical morphism $HTX + HTY + Y \longrightarrow H(TX + TY) + Y$. The homomorphism $[\![\alpha]\!]\colon TX+TY \longrightarrow TY$ has the right-hand component $[\![\alpha]\!]\,\text{inr}\colon TY \longrightarrow TY$ of which it is easy to verify that this is an endomorphism of TY in $\mathbf{Coalg}(H(-) + Y)$. Since TY is terminal, we conclude

$$[\![\alpha]\!]\,\text{inr} = \text{id}\,.$$

Thus, $[\![\alpha]\!]$ is fully determined by

$$\hat{s} \stackrel{\text{def}}{=} [\![\alpha]\!]\,\text{inl}\colon TX \longrightarrow TY.$$

Since $[\![\alpha]\!]$ is a homomorphism, one can readily check

$$s = \hat{s} \cdot \eta_X \qquad \text{and} \qquad \tau_Y \cdot H\hat{s} = \hat{s} \cdot \tau_X.$$

■

3.24. Remark. Recall the concept of a *monad* in the form of “Kleisli triples”: a monad is an endofunctor T together with a natural transformation $\eta\colon \text{Id} \longrightarrow T$ and a function assigning to every morphism $s\colon X \longrightarrow TY$ a morphism $\hat{s}\colon TX \longrightarrow TY$ such that (i) $\hat{s}\cdot\eta_X = s$, (ii) $\hat{\eta}_X = \text{id}_{TX}$ and (iii) given $t\colon Y \longrightarrow TZ$ then $\hat{t}\cdot\hat{s} = \widehat{(\hat{t}\cdot s)}$.

It follows immediately from the Substitution Theorem that T is a monad. We will see in Section 7 that T is a free completely iterative monad on H.

Thus, for a polynomial functor H_Σ we get the Σ-tree monad as a free completely iterative monad on Σ. This was originally proved in [EBT].

4. Constructions of Coalgebras

4.1. Assumption. Throughout this section H denotes an endofunctor of a category $\mathscr{A}$. The forgetful functor of $\mathbf{Coalg}\, H$ is denoted by $U\colon \mathbf{Coalg}\, H \longrightarrow \mathscr{A}$.

4.2. Colimits of Coalgebras. They are easy to construct (for the same reason for which limits of algebras are). For example, a coproduct of coalgebras $A_t \xrightarrow{\alpha_t} HA_t$, $t \in T$, is a coproduct $\coprod_{t\in T} A_t$ (with injections in_t) in $\mathscr{A}$ equipped with the unique dynamics turning each in_t into a homomorphism, viz

$$\big[H\,\mathrm{in}_t \cdot \alpha_t\big]_{t\in T}\colon \coprod_{t\in T} A_t \longrightarrow H\Big(\coprod_{t\in T} A_t\Big).$$

More generally, the following is easy to prove:

4.3. Proposition. *The forgetful functor $U\colon \mathbf{Coalg}\, H \longrightarrow \mathscr{A}$ creates colimits. That is, a colimit of a diagram $D\colon \mathscr{D} \longrightarrow \mathbf{Coalg}\, H$ is obtained from a colimit $C = \mathrm{colim}\, U{\cdot}D$ in $\mathscr{A}$ (assuming this exists) by equipping C with the unique dynamics on C turning the colimit cocone into a cocone of homomorphisms.*

4.4. Remark. (a) Limits of coalgebras are less obvious. Except those that H preserves: it is easy to see that U creates all such limits. We prove a criterion for completeness of $\mathbf{Coalg}\, H$ below.

(b) For the power-set functor $\mathscr{P}$ we know that $\mathbf{Coalg}\, \mathscr{P}$ is not complete because it does not have a terminal object (see 3.8(iv)).

(c) If H is a covarietor, then U is comonadic—this is just the dual of 3.12(d). Consequently, the existence of equalizers in $\mathbf{Coalg}\, H$ implies that $\mathbf{Coalg}\, H$ is complete: see Linton's proof of the dual statement in [Li]. That proof provides an explicit description of products in $\mathbf{Coalg}\, H$ via equalizers: let (A_i, α_i) be coalgebras, $i \in I$, and form the unique homomorphisms

$$\bar{\alpha}_i\colon A_i \longrightarrow \big(A_i\big)_{\#} \qquad \text{with } \varepsilon_{A_i} \cdot \bar{\alpha}_i = \mathrm{id}_{A_i}.$$

For the products $A = \prod A_i$ and $B = \prod (A_i)_{\#}$ in $\mathscr{A}$ we obtain a canonical morphism $k\colon A_{\#} \longrightarrow B$ extending to a homomorphism $k'\colon A_{\#} \longrightarrow B_{\#}$. And the product $\bar{\alpha} = \prod_{i\in I} \bar{\alpha}_i\colon A \longrightarrow B$ yields another homomorphism $\bar{\alpha}_{\#}\colon A_{\#} \longrightarrow B_{\#}$. The equalizer of this parallel pair

$$E \xrightarrow{\;e\;} A_{\#} \underset{\bar{\alpha}_{\#}}{\overset{k'}{\rightrightarrows}} B_{\#}$$

in $\mathbf{Coalg}\, H$ is a product of the given coalgebras (with projections $E \xrightarrow{e} A_{\#} \xrightarrow{\varepsilon_A} A \xrightarrow{\pi_i} A_i$).

4.5. Quotient Coalgebras. Recall that in General Algebra monomorphisms are precisely the homomorphisms that are carried by monomorphisms in the base category. Thus the concept of subalgebra needs no discussion: one works with the usual categorical concept of subobject. Dually, *quotient coalgebras* are just represented by epimorphisms in $\mathbf{Coalg}\, H$. This is substantiated by the following

4.6. Lemma. *Epimorphisms in* $\mathbf{Coalg}\,H$ *are precisely the homomorphisms carried by epimorphisms in* $\mathscr{A}$*.*

Proof. If a homomorphism $h\colon (A,\alpha) \longrightarrow (B,\beta)$ is an epimorphism in $\mathscr{A}$, then it is an epimorphism in $\mathbf{Coalg}\,H$ because the forgetful functor $U\colon\ \mathbf{Coalg}\,H \longrightarrow \mathscr{A}$ is faithful. For the converse, use the fact that U creates colimits: if h is an epimorphism, then a pushout of h and h (in $\mathbf{Coalg}\,H$, thus, in $\mathscr{A}$) is formed by $\mathrm{id}_B, \mathrm{id}_B$. ∎

4.7. Remark. If $\mathscr{A} = \mathbf{Set}$, then every quotient coalgebra of a coalgebra (A,α) can be represented by a *congruence*, i.e., an equivalence relation $\sim$ on A such that the canonical morphism $c\colon A \longrightarrow A/{\sim}$ is a homomorphism

$$\begin{array}{ccc} A & \xrightarrow{\ \alpha\ } & HA \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle Hc} \\ A/{\sim} & \xrightarrow[\alpha^*]{} & H(A/{\sim}) \end{array}$$

(for a, necessarily unique, α^*). And conversely, every congruence yields a quotient coalgebra on $A/{\sim}$.

4.8 *Monomorphisms* in $\mathbf{Coalg}\,H$ are more difficult (dually to the difficulties with epimorphisms well known from General Algebra). It is clear that every homomorphism carried by a monomorphism in $\mathscr{A}$ is a monomorphism in $\mathbf{Coalg}\,H$ (since U is faithful), but not conversely. In fact, if $\mathscr{A} = \mathbf{Set}$, then homomorphisms which are injective maps (i.e., are monomorphisms in $\mathscr{A}$) are precisely the regular monomorphisms of $\mathbf{Coalg}\,H$, see Theorem 3.4 in [GS2]; that paper is also the source of the following example inspired by [AM]:

4.9. Example. Denote by

$$(-)^3_2\colon\ \mathbf{Set} \longrightarrow \mathbf{Set}$$

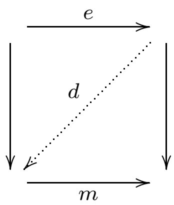
the functor which is the subfunctor of the polynomial functor $X \longmapsto X^3$ on all triples $(x_1, x_2, x_3) \in X^3$ whose coordinates are not pairwise distinct. Let A be the coalgebra on $\{a, b\}$ whose structure map $\alpha\colon A \longrightarrow A^3_2$ is defined by

$$\alpha(a) = (a, b, b) \qquad \text{and} \qquad \alpha(b) = (b, b, a).$$

It is easy to see that the (constant) map from A to the terminal, one-element, coalgebra is a monomorphism.

4.10. Subcoalgebras. Analogously to General Algebra, where the canonical factorization structure for morphisms is (regular epi, mono)—or, equivalently, (strong epi, mono), for coalgebra we propose to use the factorization system (epi, strong mono) because, as we prove below, strong monomorphisms in $\mathbf{Coalg}\,H$ are precisely those carried by strong monomorphisms in $\mathscr{A}$.

Recall that a monomorphism m is called *strong* in a category iff for every epimorphism e and every commutative square



(diagonal fill-in)

there exists a diagonal d making both triangles commutative. All "everyday" categories have (epi, strong mono)-factorization of morphisms. E.g. every *strongly complete* category, i.e., a category which has all small limits and all intersections of subobjects (possibly large), has (epi, strong mono) factorizations, see 14.21 and 14.C in [AHS].

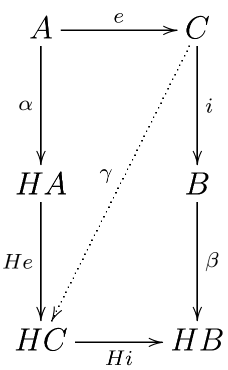
By a *subcoalgebra* of a coalgebra (A, α) we understand a strong subobject in $\mathbf{Coalg}\, H$, i.e., one represented by a strong monomorphism with codomain (A, α). This is substantiated by the following

4.11. Lemma. *If $\mathscr{A}$ has (epi, strong mono)-factorizations and H preserves strong monomorphisms, then strong monomorphisms in* $\mathbf{Coalg}\, H$ *are precisely the morphisms carried by strong monomorphisms in $\mathscr{A}$.*

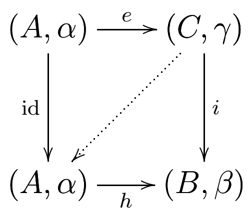
4.12. Corollary. *For endofunctors H of* $\mathbf{Set}$*, strong monomorphisms in* $\mathbf{Coalg}\, H$ *are precisely the one-to-one homomorphisms.*

Proof of 4.11 and 4.12. If a homomorphism $h\colon (A, \alpha) \longrightarrow (B, \beta)$ is a strong monomorphism in $\mathscr{A}$, then it is a strong monomorphism $\mathbf{Coalg}\, H$ due to Lemma 4.6.

Conversely, if h is a strong monomorphism in $\mathbf{Coalg}\, H$, factorize it in $\mathscr{A}$ as an epimorphism $e\colon A \longrightarrow C$ followed by a strong monomorphism $i\colon C \longrightarrow B$. Use the diagonal fill-in in $\mathscr{A}$ to make e and i homomorphisms:



By using the diagonal fill-in in **Coalg** H:



conclude that e is an isomorphism—thus, h is a strong monomorphism in $\mathscr{A}$.

For $\mathscr{A} = \mathbf{Set}$ define H' by $H'X = HX$ if $X \neq \emptyset$ and $H'\emptyset = \emptyset$ to obtain a functor H' preserving (strong) monomorphisms. It is clear that **Coalg** H' and **Coalg** H are isomorphic categories. Apply the above to H'. ∎

4.13. Corollary. *Let $\mathscr{A}$ have (epi, strong mono)-factorizations. If $\mathscr{A} = \mathbf{Set}$ or if H preserves strong monomorphisms, then* **Coalg** H *also has (epi, strong mono)-factorizations. They are, in fact, created by the forgetful functor U.*

4.14. Limit Theorem. *For every covarietor on a strongly complete category preserving strong monomorphisms the category of coalgebras is complete.*

Proof. Due to 4.4(c) it is sufficient to prove the existence of equalizers in **Coalg** H. Given homomorphisms $h, k\colon (A,\alpha) \longrightarrow (B,\beta)$ with an equalizer $e\colon E \longrightarrow A$ in $\mathscr{A}$, we denote by S the (possibly large) poset of all strong subobjects of A in $\mathscr{A}$. Since $\mathscr{A}$ is strongly complete, S is a complete lattice. For every strong subobject $m\colon M \longrightarrow A$ we have a strong subobject $Hm\colon HM \longrightarrow HA$ and pulling it back along α yields a strong subobject $\alpha^{-1}(Hm)$ in S. Put

$$m^* = \alpha^{-1}(Hm) \cap e$$

to obtain an endomap $(-)^*$ of S. Since this endomap is order-preserving, it has a largest post-fixed point $m_0\colon M_0 \longrightarrow A$ (by Knaster-Tarski theorem). Since $m_0 \subseteq \alpha^{-1}(Hm_0)$, there exists a dynamics $M_0 \longrightarrow HM_0$ turning m_0 into a homomorphism. Since $m_0 \subseteq e$, we have $hm_0 = km_0$. Thus m_0 is a equalizer of h and k. In fact, due to Corollary 4.12 we only need to show, for every homomorphism $m\colon (M,\gamma) \longrightarrow (A,\alpha)$ carried by a strong monomorphism m in $\mathscr{A}$, that

$$hm = km \qquad \text{implies} \qquad m \subseteq m_0.$$

Now $m \subseteq \alpha^{-1}(Hm)$, since m is a homomorphism, and $m \subseteq e$, since $hm = km$—thus, m is a post-fixed point of $(-)^*$. Consequently, $m \subseteq m_0$. ∎

4.15. Corollary. *For every covarietor in* **Set** *the category of coalgebras is complete.*

In fact, if H preserves strong monomorphisms, this follows from 4.14. If H does not, use the argument of 4.12 (see the end of proof).

4.16. Remark. (i) The above theorem improves a bit Corollary II.5 of [A1] where instead of strong completeness we assumed completeness and well-poweredness.

(ii) The assumption that H preserve monomorphisms is essential: a noncomplete category **Coalg** H for a covarietor on a strongly complete category is presented in [A1] (the dual of Example III.2 there).

(iii) The proof of 4.14 constructs a greatest subcoalgebra in the sense of B. Jacobs [J]. Thus, the result can also be deduced from Corollary 4.3 in [J], but the present proof is simpler. Previous results on limits of coalgebras due to J. Worrell [W2] and P. Gumm and T. Schöder [GS1] are special cases of Theorem 4.14.

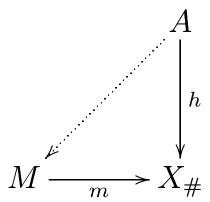
4.17. Coequations. Recall from [BH] that (systems of) equations in General Algebra can be modelled by forming a strong quotient $e\colon X^{\#} \longrightarrow E$ of (the underlying object of) a free algebra $X^{\#}$. Then an algebra A satisfies e iff every homomorphism $f\colon X^{\#} \longrightarrow A$ factorizes through e.

Dually, J. Rutten [R1] introduced presentations of classes of coalgebras over **Set** by subobjects of cofree coalgebras, for all accessible endofunctors of **Set**. There is an immediate generalization:

4.18. Definition. *Let $H\colon \mathscr{A} \longrightarrow \mathscr{A}$ be a covarietor. By a* ***coequation*** *we mean a strong subobject (in $\mathscr{A}$) of a cofree coalgebra:*

$$m\colon M \longrightarrow X_{\#} \qquad \textit{for } X \in \mathscr{A}.$$

A coalgebra A is said to ***satisfy*** *the coequation if every homomorphism $h\colon A \longrightarrow X_{\#}$ factorizes through m in $\mathscr{A}$*



4.19. Example. For deterministic systems with binary input and deadlocks, i.e., $HX = X \times X + 1$ in **Set** (see Example 2.6) recall that the terminal coalgebra $1_{\#}$ is the coalgebra of all binary trees. Let $m\colon M \hookrightarrow 1_{\#}$ be the set of all finite trees. Then a coalgebra satisfies m iff every run in the system ends in a deadlock.

For the trivial one-node tree $t \in 1_{\#}$ the coequation

$$1_{\#} - \{t\} \hookrightarrow 1_{\#}$$

is satisfied by precisely all systems without deadlocks.

4.20. Birkhoff's Covariety Theorem. *Let $H\colon$ **Set** $\longrightarrow$ **Set** be a covarietor. Then a full subcategory of* **Coalg** H *is presentable by coequations iff it is closed under coproducts, subalgebras and quotient coalgebras in* **Coalg** H.

This theorem, proved in [AP1], is a slight generalization of Rutten's original formulation in [R1]. A different approach based on dualizing Lawvere's algebraic theories, was presented by A. Kurz and J. Rosický [KR]. The relationship between the results of [KR] and [AP1] is made clear in [A2].

Birkhoff's Covariety Theorem actually holds for all endofunctors of **Set**, not necessarily varietors, as proved in [A2]—however, a somewhat subtle analysis of the concept of coequation is needed then.

4.21. Remark. Some results useful in coalgebra request that the endofunctor H should *weakly preserve pullbacks*, i.e., map pullback squares to weak pullback squares (where the latter have the factorization property of pullbacks, except that the uniqueness of factorizations is not required). This assumption is relatively weak, e.g. in $\mathscr{A} = \mathbf{Set}$ the collection of all endofunctors weakly preserving pullback contains

(a) all polynomial endofunctors,

(b) the power-set functor,

and

(c) any product, coproduct, power or composite of functors weakly preserving pullbacks.

Consequently, the functors $(-)^\Sigma \times \mathbf{bool}$, used for automata, see 2.1, and $\mathscr{P}(\Sigma \times -)$, used for labeled transition systems, see 2.3, weakly preserve pullbacks. In contrast, $(-)^3_2$, see 4.9, does not weakly preserve pullbacks.

An example of consequences of the above property we recall that if an endofunctor H weakly preserves pullbacks, then every monomorphism in $\mathbf{Coalg}\, H$ is strong, even regular, see [R1].

5. Bisimulation and Coinduction

5.1. Motivating Example. Consider labeled transition systems as coalgebras of $\mathscr{P}(\Sigma \times -)$, see 2.3. The concept of (strong) *bisimulation* goes back to R. Milner [M]: it is an equivalence between states "based intuitively on the idea that we wish to distinguish between two states if the distinction can be detected by an external agent interacting with each of them". Formally, a state a_1 in a labeled transition system A_1 is *bisimilar* to a state a_2 in A_2 provided that there exists a relation R between the state sets A_1 and A_2 such that

(i) a_1 is related to a_2, i.e., $a_1 \; R \; a_2$

(ii) for every related pair $b_1 \; R \; b_2$ and every transition $b_1 \xrightarrow{s} b_1'$ in A_1 there exists a transition $b_2 \xrightarrow{s} b_2'$ in A_2 with $b_1' \; R \; b_2'$

and

(iii) for every related pair $b_1 \; R \; b_2$ and every transition $b_2 \xrightarrow{s} b_2'$ in A_2 there exists a transition $b_1 \xrightarrow{s} b_1'$ in A_1 with $b_1 \; R \; b_1'$.

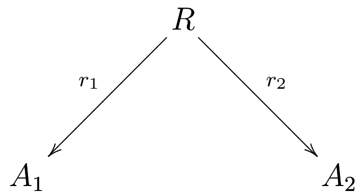
This is quite natural—and rather clumsy. The conditions (ii) and (iii) can be elegantly summarized by saying that there is a dynamics on the relation R, i.e., a function

$$R \xrightarrow{\varrho} \mathscr{P}(\Sigma \times R)$$

for which both projections $r_i \colon R \longrightarrow A_i$ $(i = 1, 2)$ become coalgebra homomorphism. The equivalence between this and (ii) & (iii) above follows from the description of coalgebra homomorphism in 2.3.

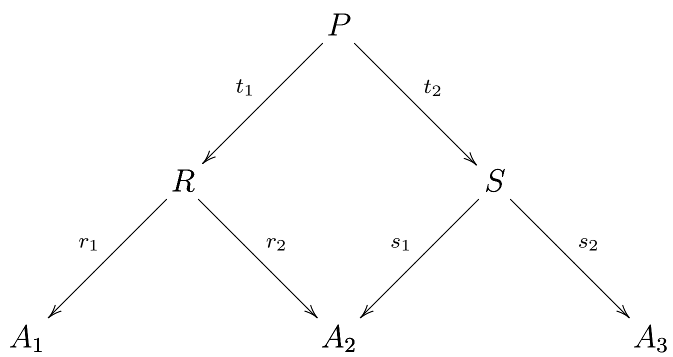
5.2. Remark. (1) In the category of sets the above example led P. Aczel [Ac] to define a bisimulation between two H-coalgebras A_1 and A_2 as a relation $R \subseteq A_1 \times A_2$ for which there exists a dynamics $R \longrightarrow HR$ turning both projections $r_i \colon R \longrightarrow A_i$ $(i = 1, 2)$ into homomorphisms of H-coalgebras.

(2) In a general finitely complete category with (epi, strong mono)-factorizations (see 4.10 for an explanation of this factorization system), a *relation* between objects A_1 and A_2 is a strong subobject of $A_1 \times A_2$. This is represented by a strong monomorphism $R \rightarrowtail A_1 \times A_2$, or, equivalently, by a pair of morphisms



with the property that the induced morphism $\langle r_1, r_2 \rangle \colon R \longrightarrow A_1 \times A_2$ is a strong monomorphism.

(3) Recall that relations are ordered (as strong subobjects of $A_1 \times A_2$) and can be composed by applying pullbacks as follows: given a relation S between A_2 and A_3 (represented by $s_1 \colon S \longrightarrow A_2$ and $s_2 \colon S \longrightarrow A_3$), form a pullback



of r_2 and s_1. Factorize $\langle r_1 t_1, s_2 t_2 \rangle \colon P \longrightarrow A_1 \times A_3$ as an epimorphism followed by a strong monomorphism, then the latter represents the composite $R \circ S$.

(4) For every relation $r_i\colon R_i \longrightarrow A_i$, $i = 1, 2$, between A_1 amd A_2 the *opposite relation* R^{-1} between A_2 and A_1 is represented by r_2 and r_1.

(5) A relation R between A and A is called an *equivalence relation* if it is reflexive (i.e., it contains the diagonal of A), symmetric (i.e., $R = R^{-1}$) and transitive (i.e., $R = R \circ R$).

5.3. Definition. *By a* ***bisimulation*** *between H-coalgebras (A_1, α_1) and (A_2, α_2) is meant a relation $r_i\colon R \longrightarrow A_i$ $(i = 1, 2)$ such that exists a dynamics on R making both r_1 and r_2 homomorphisms of H-coalgebras.*

5.4. Examples. (1) The diagonal relation on A_1, represented by $\mathrm{id}, \mathrm{id}\colon A_1 \longrightarrow A_1$, is a bisimulation between (A_1, α_1) and itself.

(2) If R is a bisimulation between A_1 and A_2, then the opposite relation R^{-1} is a bisimulation between A_2 and A_1.

(3) If H weakly preserves pullbacks, then a composite of bisimulations is a bisimulation, see [R1]. In fact, for $\mathscr{A}$ = **Set**, nonempty pullbacks are sufficient—and the converse also holds: if for an endofunctor H bisimulations on H-coalgebras are closed under composition, then H weakly preserves nonempty pullbacks, see [GS2].

5.5. Proposition. *Let $\mathscr{A}$ be a well-powered, complete category with coproducts, and let H be an endofunctor preserving strong monomorphisms. Then for every pair of coalgebras there exists a largest bisimulation between them.*

Proof. Given algebras (A_1, α_1) and (A_2, α_2), since $A_1 \times A_2$ has only a set of subobjects, there exists a set of representatives:

$$A_1 \xleftarrow{\;r_1^t\;} R^t \xrightarrow{\;r_2^t\;} A_2 \qquad (t \in T)$$

of all bisimulations.

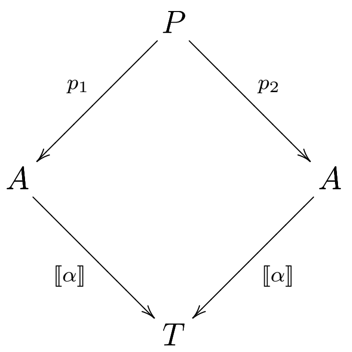
We form a coproduct $R = \coprod_{t\in T} R_t$ in $\mathscr{A}$ and obtain morphisms $r_i = [r_i^t]_{t\in T}\colon R \longrightarrow A_i$ for $i = 1, 2$. There exists a dynamics $\varrho\colon R \longrightarrow HR$ turning r_1 and r_2 into a homomorphisms (in fact, use the dynamics that exist on each R_t). Factorize $\langle r_1, r_2\rangle\colon R \longrightarrow A_1 \times A_2$ into an epimorphism followed by a strong subobject $\overline{R}$ of $A_1 \times A_2$. Then $\overline{R}$ is a bisimulation due to 4.13. Obviously, it contains each R_t, i.e., it is the largest bisimulation. ∎

5.6. Corollary. *If H moreover preserves weak pullbacks, then on every coalgebra there exists a largest bisimulation which is an equivalence relation.*

In fact, reflexivity follows from 5.4(1), symmetry from 5.4(2), and transitivity from 5.4(3).

5.7. Definition. *The largest bisimulation on a given coalgebra is called the* ***bisimilarity***.

5.8. Remark. (i) Let H weakly preserve pullbacks and have a terminal coalgebra T. For every coalgebra (A, α) we form the *kernel equivalence* of the unique homomorphism $[\![\alpha]\!]: A \longrightarrow T$, i.e., the pullback



It is easy to see that P is a relation, i.e., $\langle p_1, p_2 \rangle$ is a strong monomorphism. And since H weakly preserves pullbacks, P is a bisimulation. It follows easily that this is the bisimilarity relation:

$$\text{bisimilarity} = \text{kernel of the unique homomorphism } [\![\alpha]\!].$$

(ii) The condition of weakly preserving pullbacks is not needed here: it is sufficient to assume that H weakly preserves kernel pairs of nonempty maps. For $\mathscr{A} = \mathbf{Set}$ this last condition is equivalent to stating that every congruence (see 4.7) of every coalgebra is a bisimulation (as proved in [GS2]). This is not the case in general:

5.9. Example. (See [AM].) For the coalgebra A of Example 4.9 the largest equivalence relation, i.e., $A \times A$, is a congruence: it is the kernel equivalence of the unique homomorphism $[\![\alpha]\!]$. However, this is not a bisimulation.

5.10. Example. (i) For labeled transition systems the above definition of bisimilarity agrees with that of Example 5.1: given a span of homomorphisms $r_i: R \longrightarrow A_i$, $i = 1, 2$, then the relation of all $(r_1 x, r_2 x)$, $x \in R$, fulfils (ii) and (iii).

(ii) For $HX = X + 1$ (see 3.8), given partial unary algebras A_1 and A_2, then elements $a_1 \in A_1$ and $a_2 \in A_2$ are bisimilar iff the final homomorphisms $[\![\alpha]\!]: A_i \longrightarrow \overline{\mathbb{N}}$ fulfils

$$[\![\alpha_1]\!] a_1 = [\![\alpha_2]\!] a_2.$$

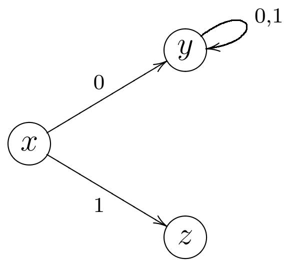
That is, iff the operations $\alpha_i: A_i \longrightarrow A_i + 1$ fulfil:

$$\alpha_1^n(a_1) \text{ is defined} \iff \alpha_2^n(a_2) \text{ is defined}$$

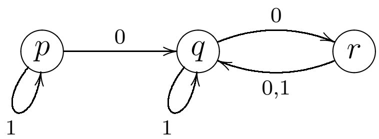
for all $n = 1, 2, 3, \ldots$

(iii) For $HA = A \times A + 1$ we consider two deterministic systems, A_1 and A_2, having inputs $0, 1$ and deadlock states. States $a_1 \in A_1$ and $a_2 \in A_2$ are bisimilar iff the unfoldings of a_1 and a_2 are equal. Example:

A_1



A_2



The states y and q are bisimilar, but no state of A_2 is bisimilar to x.

(iv) A terminal coalgebra of

$$\mathscr{P}_f$$

(the finite power-set functor, see 2.7) has been described by M. Barr [B2] as the quotient coalgebra of the coalgebra of all finitely branching trees modulo bisimilarity. A simpler description is due to J. Worrell [W1]: A nonordered finitely branching tree is called *strongly extensional* if two children of one parent defining bisimilar subtrees are always equal. (Recall that trees are considered up-to tree isomorphism, and being $\mathscr{P}_f$-coalgebras, see 2.7, we have the concept of bisimilarity for them.)

The set T of all finitely branching, strongly extensional trees has a natural dynamics $\alpha \colon T \longrightarrow \mathscr{P}_f T$ assigning to every tree the set of all maximum proper subtrees. This coalgebra is terminal.

5.11. Corecursion. This is a method of constructing morphisms from an object A into a terminal coalgebra T by equipping A with a "suitable" dynamics $\alpha \colon A \longrightarrow HA$ for which the desired morphism is $[\![\alpha]\!] \colon A \longrightarrow T$.

The following examples stem from [R1].

5.12. Example. Corecursive definition of addition. Recall the terminal coalgebra $\overline{\mathbb{N}}$ of $HX = X + 1$, 3.8(i), and search for a dynamics

$$\mathrm{add} \colon \overline{\mathbb{N}} \times \overline{\mathbb{N}} \longrightarrow \overline{\mathbb{N}} \times \overline{\mathbb{N}} + 1$$

such that

$$[\![\mathrm{add}]\!] \colon \overline{\mathbb{N}} \times \overline{\mathbb{N}} \longrightarrow \overline{\mathbb{N}}$$

is the usual addition + of natural numbers extended with ∞. That is, such that the following square, where $*$ is the unique element of 1,

$$\begin{array}{ccc} \overline{\mathbb{N}} \times \overline{\mathbb{N}} & \xrightarrow{\text{add}} & \overline{\mathbb{N}} \times \overline{\mathbb{N}} \cup \{*\} \\ {\scriptstyle (+)}\downarrow & & \downarrow{\scriptstyle (+)\cup \mathrm{id}_1} \\ \overline{\mathbb{N}} & \xrightarrow[\tau]{} & \overline{\mathbb{N}} \cup \{*\} \end{array}$$

commutes. That is, for $(n, m) \neq (0, 0)$ the result (n', m') of add fulfils $n+m-1 = n'+m'$. It is not difficult to see that the following dynamics "works":

$$\mathrm{add}(n, m) = \begin{cases} (n-1, m) & \text{if } n > 0; \\ (0, m-1) & \text{if } n = 0 \text{ and } m > 0; \\ * & \text{if } m = n = 0. \end{cases}$$

5.13. Example. Corecursive definitions of infinite streams. Here we use the terminal coalgebra Σ^ω of $HX = \Sigma \times X$, see 3.8(ii). For every symbol $a \in \Sigma$ we can define the element a^ω of Σ^ω (the stream of all a's) by using the obvious dynamics on the set 1:

$$a\colon 1 \longrightarrow \Sigma \cong \Sigma \times 1.$$

Then

$$[\![a]\!]\colon 1 \longrightarrow \Sigma^\omega$$

is the desired stream. In fact, since the following square

$$\begin{array}{ccc} 1 & \xrightarrow{a} & \Sigma^\omega \\ {\scriptstyle [\![\alpha]\!]}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \times [\![a]\!]} \\ \Sigma^\omega & \xrightarrow[\langle \mathrm{head}, \mathrm{tail} \rangle]{} & \Sigma \times \Sigma^\omega \end{array}$$

commutes, we have

$$\mathrm{head}[\![a]\!] = a$$

and

$$\mathrm{tail}[\![a]\!] = [\![a]\!]$$

which is precisely (a coinductive) definition of a^ω.

How can we define e.g. $(a\,b\,a\,b\,a\,b\ldots) = (ab)^\omega$?

Here we use a dynamics on $1 + 1$:

$$a + b\colon 1 + 1 \longrightarrow \Sigma + \Sigma \cong \Sigma \times (1 + 1).$$

Then

$$[\![a + b]\!]\colon 1 + 1 \longrightarrow \Sigma^\omega$$

is easily seen to have components $(ab)^\omega$ and $(ba)^\omega$, respectively.

5.14. Example. Corecursive definition of zipping. We want to define the binary operation on Σ^ω which zips two streams v and w into one by taking successively an element of v, then an element of w, then one of v etc. For that, we want to find a dynamics

$$\mathrm{zip}\colon \Sigma^\omega \times \Sigma^\omega \longrightarrow \Sigma \times \Sigma^\omega \times \Sigma^\omega$$

with the property that

$$[\![\mathrm{zip}]\!]\colon \Sigma^\omega \times \Sigma^\omega \longrightarrow \Sigma^\omega$$

is the desired zipping operation.

The general strategy for describing α from $[\![\alpha]\!]$ is: α does the first step which $[\![\alpha]\!]$ infinitely repeats. This suggests to try

$$\mathrm{zip}(v,w) = (\mathrm{head}\, v, w, \mathrm{tail}\, v)$$

or, more succinctly (using projections π_1 and π_2 of $\Sigma^\omega \times \Sigma^\omega$):

$$\mathrm{zip} = \langle \mathrm{head} \cdot \pi_1, \pi_2, \mathrm{tail} \cdot \pi_1 \rangle.$$

In fact, since $[\![\mathrm{zip}]\!]$ is a coalgebra homomorphism we have that

$$\mathrm{head}\big([\![\mathrm{zip}]\!](v,w)\big) = \mathrm{head}\, v$$

and

$$\mathrm{tail}\big([\![\mathrm{zip}]\!](v,w)\big) = [\![\mathrm{zip}]\!](w, \mathrm{tail}\, v)$$

which is the desired operation.

5.15. Coinduction. This is a method of proofs using the fact that no two states of a terminal coalgebra $T \xrightarrow{\tau} HT$ are bisimilar. Thus, in order to verify the equality of two states of T, it is sufficient to find a bisimulation on T under which they are related.

5.16. Example. Commutativity of addition. We want to prove

$$[\![\mathrm{add}]\!](n,m) = [\![\mathrm{add}]\!](m,n).$$

We are looking for a bisimulation R on the terminal coalgebra $\overline{\mathbb{N}}$ of $HA = A + 1$ with

$$[\![\mathrm{add}]\!](n,m)\; R\; [\![\mathrm{add}]\!](m,n).$$

In fact, we will find a bisimulation equivalence $\sim$ which means that the canonical epimorphism $e\colon \overline{\mathbb{N}} \longrightarrow \overline{\mathbb{N}}/{\sim}$ carries a homomorphism

$$\begin{array}{ccc}
\overline{\mathbb{N}} & \xrightarrow{\quad\tau\quad} & \overline{\mathbb{N}} + 1 \\
{\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e+\mathrm{id}_1} \\
\overline{\mathbb{N}}/{\sim} & \xrightarrow[\quad\alpha\quad]{} & \overline{\mathbb{N}}/{\sim} + 1
\end{array}$$

(for some α). This is equivalent to the statement that if $i \sim j$ then $\tau i \sim \tau(j)$, i.e., if one of i, j is zero, then both are, else, $i - 1 \sim j - 1$. From the coinductive definition of $[\![\text{add}]\!]$ we get

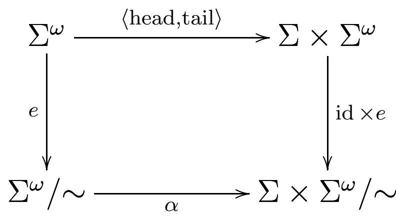
$$[\![\text{add}]\!](n,m) - 1 = \begin{cases} [\![\text{add}]\!](n-1,m) & \text{if } n > 0; \\ [\![\text{add}]\!](0,m-1) & \text{if } n = 0,\ m > 0 \end{cases}$$

and we conclude that the smallest equivalence relation $\sim$ with $[\![\text{add}]\!](n,m) \sim [\![\text{add}]\!](m,n)$ for all m, n is indeed a bisimulation of $\overline{\mathbb{N}}$. This proves that $\sim$ is the identity relation, i.e., that $[\![\text{add}]\!]$ is commutative.

5.17. Example. The equality $[\![\text{zip}]\!](a^\omega, b^\omega) = (ab)^\omega$. We try to define a bisimulation equivalence $\sim$ on the terminal coalgebra Σ^ω satisfying

(2) $$[\![\text{zip}]\!](a^\omega, b^\omega) \sim (ab)^\omega.$$

To say that $\sim$ is a bisimulation equivalence means that the following square



commutes for some α. Equivalently: given streams $v \sim w$ then

$$\text{head}\, v = \text{head}\, w$$

and

$$\text{tail}\, v \sim \text{tail}\, w.$$

¿From the coinductive definitions of a^ω, b^ω, $(a,b)^\omega$ and zip it clearly follows that

$$\text{head}[\![\text{zip}]\!](a^\omega, b^\omega) = a = \text{head}(ab)^\omega,$$

and

$$\text{tail}[\![\text{zip}]\!](a^\omega, b^\omega) = [\![\text{zip}]\!](b^\omega, a^\omega),$$

$$\text{tail}(a,b)^\omega = (ba)^\omega.$$

Thus, let $\sim$ be the equivalence relation on Σ^ω whose only nontrivial equivalence classes are given by (2) and

(3) $$[\![\text{zip}]\!](b^\omega, a^\omega) \sim (ba)^\omega.$$

Then $\sim$ is a bisimulation equivalence. By coinduction, this proves

$$[\![\text{zip}]\!](a^\omega, b^\omega) = (ab)^\omega \qquad \text{and} \qquad [\![\text{zip}]\!](b^\omega, a^\omega) = (ba)^\omega.$$

5.18. Remark. Corecursion and coinduction have been used in a number of beautiful applications, e.g., a coinductive calculus of streams, automata and power series in [R2], equivalence of applicative structures in [HL], coinductive combinatorics in [R3], or in connection with distributive laws in [Br]. It is only due to lack of space that we refrain from giving any details of these applications.

6. Iterative Algebras

6.1. Remark. In the "classical world" of Σ-algebras in **Set** the concept of iterative algebra was introduced by Evelyn Nelson[N] in her attempt to simplify the (very complex) proof of the description of free iterative theories provided by Calvin Elgot and his collaborators [E], [BE], [EBT]. A similar concept of iterative algebra was studied by Jerzy Tiurin [T].

Recently this has been generalized to every finitary functor $H\colon \mathscr{A} \longrightarrow \mathscr{A}$ where $\mathscr{A}$ is a locally presentable category: A free iterative theory on H has been described in [AMV3] as the theory of free iterative H-algebras. We indicate the procedure here.

6.2. Example. (E. Nelson) Given a Σ-algebra (or H_Σ-algebra) of a finitary signature Σ

$$\alpha\colon H_\Sigma A \longrightarrow A$$

by a *recursive system of equations* in A is meant a system

$$\begin{aligned} x_1 &= t_1(x_1, \dots, x_n, a_1, \dots, a_k) \\ &\vdots \\ x_n &= t_n(x_1, \dots, x_n, a_1, \dots, a_k) \end{aligned} \tag{4}$$

whose right-hand sides are Σ-terms over $X + A$ where $X = \{x_1, \dots, x_n\}$ is a set of (bound) variables and the parameters $a_1, \dots, a_k$ are elements of A. A *solution* of (4) is an interpretation $x_i \longmapsto x_i^\dagger \in A$ of the variables in A in such a way that the formal equations become equalities

$$x_i^\dagger = t_i(x_1^\dagger, \dots, x_n^\dagger, a_1, \dots, a_k) \qquad (i = 1, \dots, n)$$

in A. There are equations which automatically have many solutions, e.g., $x_1 = x_1$. To exclude these, we restrict ourselves to *guarded* systems (4) which are those such that

$$\text{no } t_i \text{ is a single variable in } X \qquad (i = 1, \dots, n).$$

6.3. Definition. (See [N].) *A Σ-algebra is called **iterative** if every guarded system* (4) *of equations has a unique solution.*

6.4. EXAMPLE. Unary algebras. If Σ consists of a single unary symbol then for every iterative algebra $\sigma\colon A \longrightarrow A$ we have a unique fixed point of σ: consider the guarded recursive equation

$$x_1 = \sigma(x_1).$$

Also σ^2 has a unique fixed point (i.e., σ has no 2-cycles): consider

$$\begin{aligned} x_1 &= \sigma(x_2) \\ x_2 &= \sigma(x_1) \end{aligned}$$

etc. Conversely, it is not hard to prove that every algebra such that each of $\sigma, \sigma^2, \sigma^3, \ldots$ has a unique fixed point is iterative. See [AMV3].

6.5. EXAMPLE. Binary algebras. If Σ consists of a single binary symbol, the iterativity of an algebra $\sigma\colon A \times A \longrightarrow A$ does not have a simple description. Observe that σ has a unique idempotent, due to

$$x = \sigma(x, x).$$

Moreover, for every $a \in A$ there exists a unique $\bar{a} \in A$ with $\bar{a} = \sigma(a, \bar{a})$, due to

$$x = \sigma(a, x),$$

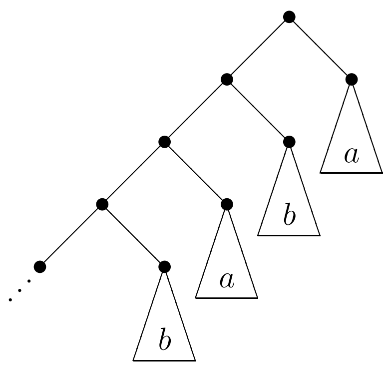
etc.

"Classical" algebras are usually not iterative. For example, an iterative group $(G, *, e)$ is trivial due to the unique solution of $x = x * e$.

An example of a nice iterative algebra is the algebra T of all (finite and infinite) binary trees. The following example demonstrates the general procedure of finding solutions of recursive systems of equations: consider

$$\begin{aligned} x_1 &= \sigma(x_2, a) \\ x_2 &= \sigma(x_1, b) \end{aligned}$$

for binary trees $a, b \in T$. The solution $x_1^\dagger$ has the right-hand child a. The left-hand child is x_2, thus, it has b as the right-hand child, etc. Here is the whole tree

(5) $x_1^\dagger =$ 

Analogously for $x_2^\dagger$.

6.6. Definition. (S. Ginali, see [Gi].) *A Σ-tree on X (see 3.16) is called* ***rational*** *if it has, up-to isomorphism, only finitely many subtrees. Denote by*

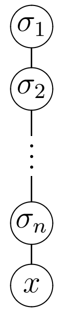
$$R_\Sigma X$$

the subalgebra of $T_\Sigma X$ of all rational trees.

6.7. Example. The tree (5) is rational whenever a and b are: all subtrees of $x_1^\dagger$ are isomorphic to subtrees of a or b, or to $x_1^\dagger$ or $x_2^\dagger$.

6.8. Proposition. (See [G], [N].) *The rational-tree algebra $R_\Sigma X$ is a free iterative algebra on X. That is, $R_\Sigma X$ is iterative and for every iterative Σ-algebra A and every function $f\colon X \longrightarrow A$ there exists a unique homomorphism $\bar{f}\colon R_\Sigma X \longrightarrow A$ extending f.*

6.9. Example. Let Σ be a unary signature. Then a free Σ-algebra on X is the algebra $\Sigma^* \times X$ of all trees



with $\sigma_1 \dots \sigma_n$ in Σ^* and $x \in X$. The algebra $T_\Sigma X = \Sigma^* \times X + \Sigma^\infty$ consists of adding to $\Sigma^* \times X$ all infinite sequences over Σ. And the rational-tree algebra $R_\Sigma X$ adds to $\Sigma^* \times X$ precisely those sequences Σ^∞ which are "eventually periodical" (i.e., periodical after removing a finite prefix).

6.10. Remark. Let us call a system of equations (4) *flat* if each of the right-hand sides $t_1, \dots, t_n$ is either a single element of A, or a *flat term*

$$\sigma(y, z, \dots) \in X^k$$

for some operation symbol $\sigma \in \Sigma_k$ and for k (not necessarily distinct) variables $y, z, \dots$ in X. E. Nelson observed in [N] that a Σ-algebra is iterative iff every flat system of equations (4) has a unique solution. This serves for us as a basis for a categorical generalization: a flat recursive equation system in A is, obviously, represented by a function

$$e\colon X \longrightarrow H_\Sigma X + A$$

where X is the (finite) set of variables. In fact, e assigns to every variable the appropriate right-hand side. This leads us to the following generalization using finite presentability (see 3.15c):

6.11. Definition. *Given an endofunctor H, an H-algebra $\alpha\colon HA \longrightarrow A$ is called* ***iterative*** *provided that every* ***finitary flat equation morphism***

$$e\colon X \longrightarrow HX + A \qquad (X \textit{ finitely presentable})$$

there exists a unique ***solution****, i.e., a unique morphism*

$$e^\dagger\colon X \longrightarrow A$$

such that the square

(6)
$$\begin{array}{ccc} X & \xrightarrow{\ e^\dagger\ } & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [\alpha, A]} \\ HX + A & \xrightarrow[He^\dagger + A]{} & HA + A \end{array}$$

commutes.

6.12. Remark. For polynomial endofunctors and flat systems

(i) guardedness is not mentioned explicitly: flat implies guarded, and

(ii) the square (6) precisely expresses the condition that the function $e^\dagger\colon x_i \longmapsto x_i^\dagger$ from X to A is a solution: if the right-hand side t_i is an element $a \in A$, then $x_i^\dagger = a$, and if t_i is a flat term $t_i = \sigma(y, z, \dots)$, then $x_i^\dagger = \sigma(y^\dagger, z^\dagger, \dots)$ means $e^\dagger(x_i) = \alpha \cdot H_\Sigma e^\dagger \cdot e(x_i)$.

6.13. Example. (See [Mo] or [AAMV].) Terminal coalgebras are iterative. More precisely, if $T \xrightarrow{\tau} HT$ is a terminal coalgebra, then $HT \xrightarrow{\tau^{-1}} T$ is an iterative algebra. Applied to $H(-) + Y$ (compare 2.9) this tells us that whenever a terminal coalgebra TY of $H(-) + Y$ exists then the corresponding H-algebra TY is iterative.

6.14. Proposition. (See [AMV3].) *Iterative algebras are closed in* $\mathbf{Alg}\, H$ *under limits and filtered colimits. Consequently, the category of iterative algebras and homomorphisms is reflective in* $\mathbf{Alg}\, H$*.*

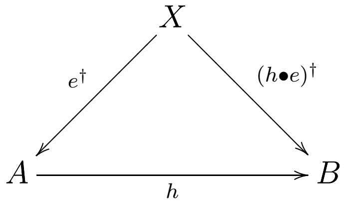
6.15. Corollary. *Free iterative algebras exist.*

In fact, since H is a finitary functor, free H-algebras exist, see [A0], in other words, the canonical forgetful functor $U\colon \mathbf{Alg}\, H \longrightarrow \mathscr{A}$ has a left adjoint. Since the embedding $E\colon \mathbf{Alg}_{\mathrm{it}}\, H \hookrightarrow \mathbf{Alg}\, H$ of the full subcategory of all iterative algebras also has a left adjoint (by 6.14), the forgetful functor $UE\colon \mathbf{Alg}_{\mathrm{it}}\, H \longrightarrow \mathscr{A}$ also has a left adjoint.

6.16. Remark. The above proposition takes all homomorphisms as the choice of morphisms for iterative algebras. The reader may wonder whether a more appropriate choice should be considered, involving "preservation of solutions". In fact, the interpretation of the latter is rather obvious: given iterative algebras $\alpha\colon HA \longrightarrow A$ and $\beta\colon HB \longrightarrow B$ and a morphism $h\colon A \longrightarrow B$, then for every flat equation morphism in A, say $e\colon X \longrightarrow HX + A$, we have the canonical equation morphism in B:

$$h \bullet e \equiv X \xrightarrow{e} HX + A \xrightarrow{HX+h} HX + B.$$

We say that h *preserves solutions* if that for every flat equation morphism $e\colon X \longrightarrow HX + A$ the triangle



commutes. Luckily, this coincides with the concept of homomorphism:

6.17. Lemma. [AMV3] *Given iterative algebras (A, α) and (B, β), then a morphism $h\colon A \longrightarrow B$ preserves solutions iff it is a homomorphism.*

6.18. Definition. *The monad of all free iterative H-algebras is called the* ***rational monad*** *of H. Notation:*

$$(R, \eta, \mu).$$

Explicitly, for every object X we form a free iterative algebra RX with the algebra structure

$$\varrho_X\colon HRX \longrightarrow RX$$

and the universal arrow

$$\eta_X\colon X \longrightarrow RX.$$

And we denote by

$$\mu_X\colon R(RX) \longrightarrow RX$$

the unique homomorphism with $\mu_X \cdot \eta_{RX} = \mathrm{id}$.

6.19. Examples. (See [A3].)

(1) For the polynomial functors H_Σ in **Set**, the rational monad R_Σ is the rational-tree monad, see 6.8.

(2) Let $\mathscr{P}_2\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ denote the functor assigning to a set X the set $\mathscr{P}_2 X$ of all non-ordered pairs in X, defined on maps $f\colon X \longrightarrow Y$ by $\mathscr{P}_2 f\colon \{x, y\} \longmapsto \{f(x), f(y)\}$. Observe that $\mathscr{P}_2$-algebras are just binary algebras whose operation is commutative.

The rational monad assigns to a set X the set RX of all nonordered rational binary trees with leaves labeled in X. This is an obvious quotient monad of the rational monad R_Σ where Σ consists of one binary operation.

(3) The rational monad of $\mathscr{P}_f$, the finite-power-set functor, is the monad of all strongly extensional (see 5.10(iv)) rational, finitely branching trees.

6.20. Remark. The above examples demonstrate the following general description of rational monads on **Set**:

Given a finitary endofunctor H there exists a finitary signature Σ such that H is a quotient of H_Σ. (For example, in case of $\mathscr{P}_f$ consider Σ having a single n-ary operation σ_n for every $n = 0, 1, 2, \dots$) There are equations between flat (see 6.10) Σ-terms such that H is obtained from H_Σ by applying these equations finitely many times—we call these equations *basic*. (Example: for $\mathscr{P}_f$ the basic equations are all the equations

$$\sigma_n(x_1, \dots, x_n) = \sigma_k(y_1, \dots, y_k)$$

where $\{x_1, \dots, x_n\} = \{y_1, \dots, y_k\}$.)

Then the rational monad R of H is obtained from the rational monad R_Σ as a quotient modulo the congruence obtained by applying the basic equations finitely or infinitely many times. See [A3] for a precise definition of infinite application of equations (and a precise proof of the above description of R).

6.21. Remark. (See [AMV3].) (1) Observe that the morphisms $\varrho_X \colon HRX \longrightarrow RX$ and $\eta_X \colon X \longrightarrow RX$ of 6.18 form natural transformations

$$\varrho\colon HR \longrightarrow R \qquad \text{and} \qquad \eta\colon \mathrm{Id} \longrightarrow R$$

with the property that R is a coproduct

$$R = HR + \mathrm{Id} \qquad \text{with injections } \varrho \text{ and } \eta.$$

(2) Every iterative Σ-algebra allows for a canonical computation of rational terms, i.e., terms expressed by rational trees. This is the special case of the following general phenomenon:

Given an iterative algebra $HA \xrightarrow{\alpha} A$, we obtain the unique homomorphism

$$\widehat{\alpha}\colon RA \longrightarrow A$$

of H-algebras with $\widehat{\alpha}\cdot\eta_A = \mathrm{id}$. Then $(A, \widehat{\alpha})$ is an Eilenberg-Moore algebra of the rational monad.

However, iterative algebras are not monadic on $\mathscr{A}$; the category of Eilenberg-Moore algebras of R is described in [AMV4] as the category of all Elgot algebras. These are algebras with specified solutions of flat equation morphisms such that the specification satisfies two (rather simple) axioms.

7. Iterative Theories

7.1. Remark. What is the connection of Section 6 to coalgebra? It turns out that there is one that is deep, technically difficult to prove, and useful:

7.2. Theorem. (See [AMV3].) *Let H be a finitary endofunctor of a locally finitely presentable category. Then the initial iterative algebra, RO, can be constructed as a filtered colimit of all H-coalgebras on finitely presentable objects.*

7.3. Corollary. *Let Eq_Y be the diagram of all coalgebras of $H(-) + Y$ on finitely presentable objects (that is, all finitary, flat equation morphisms $e\colon X \longrightarrow HX + Y$). Then the objectwise description of the rational monad on H is*

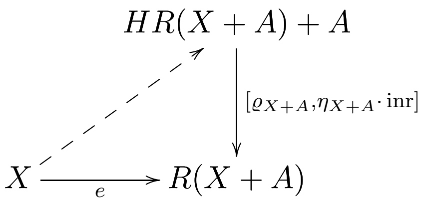
$$RY = \operatorname{colim} Eq_Y .$$

7.4. Remark. One consequence of Theorem 7.2 is the fact that, precisely as in the case of Σ-algebras (see Remark 6.9), every iterative algebra A allows for unique solutions of non-flat equation morphisms if they are guarded.

Generalizing flat equation morphism (see 6.10), let us call a morphism

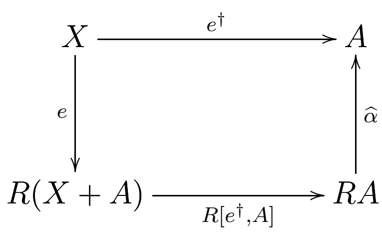
$$e\colon X \longrightarrow R(X + A) \qquad X \text{ finitely presentable}$$

a *rational equation morphism.* It is *guarded* if it factorizes through the first summand of $R(X + A) = \big[HR(X + A) + A\big] + X$ (see 6.21):



This generalizes, in case of H_Σ, the equation system (4) of 6.2 by allowing the right-hand sides to be not only terms, i.e., finite Σ-trees, but also rational Σ-trees. Now the concept of solution in Example 6.2 generalizes as follows (using the notation $\widehat{\alpha}$ of Remark 6.21):

7.5. Theorem. *For every iterative algebra $HA \xrightarrow{\alpha} A$ and every guarded rational equation morphism $e\colon X \longrightarrow R(X + A)$ there exists a unique* ***solution*** $e^\dagger\colon X \longrightarrow A$ *i.e., a unique morphism such that the square*



commutes.

7.6. Remark. In the language of algebraic theories (i.e., finitary monads on **Set**) the above Theorem together with Remark 6.21 tell us that the rational monad is an iterative algebraic theory for every finitary endofunctor of **Set**. Recall the concept of an iterative theory of C. Elgot [E] as rephrased categorically in [AAMV] for monads over locally finitely presentable categories:

(a) A monad (S, η, μ) is called *ideal* if S has a subfunctor $\sigma\colon S' \rightarrowtail S$ such that

(i) S is a coproduct $S = S' + \mathrm{Id}$ (with injections $\sigma\colon S' \longrightarrow S$ and the unit $\eta\colon \mathrm{Id} \longrightarrow S$) and

(ii) μ has a restriction to a natural transformation

$$\mu'\colon S'S \longrightarrow S' \qquad \text{with } \sigma{\cdot}\mu' = \mu{\cdot}\sigma S.$$

More precisely, the sixtuple $(S, \eta, \mu, S', \sigma, \mu)$ is called an ideal monad.

Example: let $\mathscr{A}$ be a locally finitely presentable category in which coproduct injections are monic. The rational monad $R = HR + \mathrm{Id}$ (see 6.20) is ideal w.r.t. the subfunctor $\varrho\colon HR \rightarrowtail R$ and $\mu' = H\mu$.

(b) Given an ideal monad, by a *finitary equation morphism* is meant a morphism

$$e\colon X \longrightarrow S(X + Y)$$

where X is a finitely presentable object, and Y is an arbitrary object; e is called *guarded* if it factorizes through the first summand of $S(X + Y) = [S'(X + Y) + Y] + X$.

(c) The ideal monad is called *iterative* provided that for every guarded finitary equation morphism e there exists a unique *solution* $e^\dagger$, i.e., a unique morphism $e^\dagger\colon X \longrightarrow SY$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & SY \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu_Y} \\ S(X+Y) & \xrightarrow[S[e^\dagger, \eta_Y]]{} & SSY \end{array}$$

commutes.

(d) Given an ideal monad S, by an *ideal natural transformation* from a functor H to S is called a natural transformation $H \longrightarrow S$ which factorizes through σ. Example: if R is the rational monad of H (w.r.t. $\varrho\colon HR \longrightarrow R$, see 6.21), then the canonical natural transformation κ from H to R:

$$\kappa \equiv H \xrightarrow{\;H\eta\;} HR \xrightarrow{\;\varrho\;} R$$

is ideal.

For two ideal monads $(S, \eta, \mu, S', \sigma, \mu')$ and $(\overline{S}, \overline{\eta}, \overline{\mu}, \overline{S}', \overline{\sigma}, \overline{\mu}')$ we will call a given monad morphism $h\colon (S, \eta, \mu) \longrightarrow (\overline{S}, \overline{\eta}, \overline{\mu})$ *ideal* if it has a domain-codomain restriction $h'\colon S' \longrightarrow \overline{S}'$.

7.7. Theorem. (See [AMV3].) *The rational monad R is a free iterative monad on H. That is*

(i) *R is an iterative monad,*

(ii) *for every iterative monad S and every ideal natural transformation $h\colon H \longrightarrow S$ there exists a unique ideal monad morphism $\bar{h}\colon R \longrightarrow S$ with $h = \bar{h}\cdot\kappa$.*

8. Complete Iterativity

8.1. Assumption. In this section H denotes an endofunctor of a category $\mathscr{A}$ having finite coproducts with monic injections.

The present section is distinguished from the previous two by dropping the assumption, for all equation morphisms $X \longrightarrow HX + A$ or $X \longrightarrow R(X + A)$ etc., that X be finitely presentable.

8.2. Definition. *An H-algebra $HA \xrightarrow{\alpha} A$ is* ***completely iterative*** *provided that for every flat equation morphism $e\colon X \longrightarrow HX + A$, where X is an arbitrary object, there exists a unique solution $e^\dagger\colon X \longrightarrow A$ (defined, again, by the commutativity of the square* (6) *see* 6.11*).*

8.3. Example. (See [Mo] or [AAMV].) (i) If $T \xrightarrow{\tau} HT$ is a terminal H-coalgebra, then $HT \xrightarrow{\tau^{-1}} T$ is an initial completely iterative H-algebra.

(ii) More generally: let us call H *iteratable*, i.e., for every object Y a terminal coalgebra, TY, of $H(-) + Y$ exists. It follows that TY is a coproduct

$$TY = HTY + Y \qquad \text{with injections } \tau_Y\colon HTY \longrightarrow TY,\ \eta_Y\colon Y \longrightarrow D$$

and the H-algebra (TY, τ_Y) is completely iterative.

8.4. Theorem. (See [Mi].) *If H is iteratable, then TY is a free completely iterative H-algebra on Y. Conversely, whenever free completely iterative H-algebras exist, then H is iteratable.*

8.5. Example. (See [AAMV].) If $\mathscr{A}$ is locally presentable, then every accessible endofunctor (see 3.15) is iteratable. But there exist iteratable functors which are not accessible.

8.6. Remark. We can define *completely iterative monads* precisely as in Remark 7.6, just dropping the assumption of X being finitely presentable (everywhere). Then analogously to 7.7 we obtain the result that the monad T of free completely iterative algebras is a free completely iterative monad on H; see [AAMV].

8.7. Example. For polynomial endofunctors H_Σ of **Set** the above monad is the monad T_Σ of all Σ-trees. This is the original example of a free completely iterative monad on Σ introduced in [EBT].

9. Parametrized Iterativity

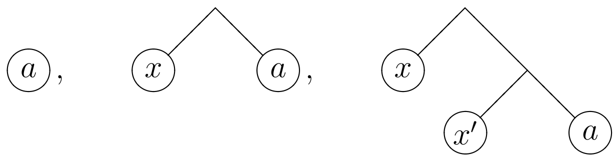
9.1. Remark. Tarmo Uustalu proposed in [U] a generalization of the environment in which iterativity is studied: in place of a finitary functor $H\colon \mathscr{A} \longrightarrow \mathscr{A}$ he considers a finitary functor $H\colon \mathscr{A} \longrightarrow FM(\mathscr{A})$,

where $FM(\mathscr{A})$ is the category of finitary monads

on $\mathscr{A}$. We briefly sketch the basic situation of Σ-algebras with parametrized iterativity which this environment is capable to represent, and then formulate the appropriate generalization of the results of Section 7.

9.2. Example. (See [AMV5].) Consider first one binary operation σ. In the original concept of iterative algebras of E. Nelson (see Example 6.5) both variables x_1 and x_2 in $\sigma(x_1, x_2)$ can be used for iteration in the equational systems (4) of Example 6.2. This leads to the rational monad R_Σ of all rational binary trees.

Now let us decree that x_1 can be used for iteration, but x_2 cannot. This corresponds to equation systems (4) where the right-hand sides t_i all have the form

(7) 

, etc

for variables $x, x', \ldots$ in X and an element $a \in A$.

Again, we call an algebra $A \times A \xrightarrow{\sigma} A$ *iterative* if every such guarded system of equations has a unique solution. This is strictly weaker than the original concept of iterativity—e.g., here an iterative algebra does not need to posses an idempotent element.

We can formalize this weaker iterativity by introducing, for every set X, the "derived" signature of all operation symbols $\sigma(x, -)$ (unary), for $x \in X$. Let us denote by

$$X \,\square\, A$$

a free algebra of this signature on A—this is precisely where the right-hand sides (7) lie. That is, the current form of finitary equation morphisms is

$$e\colon X \longrightarrow X \,\square\, A \qquad (X \text{ is finite}).$$

Now $X \,\square\, -$ is a monad of **Set**, the free-algebra monad of the derived signature given by $X \mapsto X^* \times A$. Thus $\square$ is the uncurried version of a functor from **Set** to $FM(\mathbf{Set})$ (assigning to every set X the monad $A \mapsto X \,\square\, A$). Also, every binary algebra $\sigma\colon A \times A \longrightarrow A$ defines an Eilenberg-Moore algebra

$$\hat{\sigma}\colon A \,\square\, A \longrightarrow A$$

of the monad $A \,\square\, -$: the function $\hat{\sigma}\colon A^* \times A \longrightarrow A$ computes the terms of $A \,\square\, A$ in A. (Conversely, every Eilenbderg-Moore algebra of $A \,\square\, -$ on the set A is uniquely determined by some σ.)

An algebra $\sigma\colon A \times A \longrightarrow A$ is now called *iterative* if for every finitary equation morphism $e\colon X \longrightarrow X \,\square\, A$ there exists a unique *solution* $e^\dagger\colon X \longrightarrow A$ given by the commutativity of

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & A \\ {\scriptstyle e}\big\downarrow & & \big\uparrow{\scriptstyle \hat\sigma} \\ X \,\square\, A & \xrightarrow[\;e^\dagger \square A\;]{} & A \,\square\, A \end{array}$$

A free iterative algebra in the present sense is the algebra of all rational binary trees on X which are *right-wellfounded*, i.e., the right-most path from any node is always finite.

9.3. Example. (See [AMV5].) The previous example is a special case of a *parametrized signature*, i.e., a signature Σ with an additional function assigning to every symbol σ of arity n a number

$$\mathrm{it}(\sigma) = 0, 1, \dots, n \qquad \text{(the } \textit{iterativity} \text{ of } \sigma).$$

This additional function does not play any role for the concept of Σ-algebra and homomorphism. But it influences our concept of recursive system of equations, see 6.2: we allow, for every node of a right-hand side tree t_i labeled by σ, only the first $\mathrm{it}(\sigma)$ children to be iteratable. More precisely, for (4) in 6.2 we request that

(i) every leaf of t_i labeled by a variable $x_1, \dots, x_n$ has a parent labeled by $\sigma \in \Sigma$, and the leaf is one of the first $\mathrm{it}(\sigma)$ children

and

(ii) every inner node of t_i labeled by $\sigma \in \Sigma$ has the first $\mathrm{it}(\sigma)$ children labeled by a variable $x_1, \dots, x_n$.

A Σ-algebra is called *iterative* (w.r.t. the parametrized signature) if every such guarded system of recursive equations has a unique solution.

An elegant way of getting rid of the complicated conditions (i) and (ii) is to form, for every set X of variables, the *derived signature* of all symbols

$$\sigma(x_1, \dots, x_i, -) \qquad \text{for } \sigma \in \Sigma,\ i = \mathrm{it}(\sigma) \text{ and } (x_1, \dots, x_i) \in X^i$$

having arity $\mathrm{ar}(\sigma) - i$. Denote by

$$X \,\square\, A$$

a free algebra of A of the last signature. Then (i) and (ii) precisely describe the trees that naturally form $X \,\square\, A$. In other words, a recursive system of equations is now expressed by a morphism

$$e\colon X \longrightarrow X \,\square\, A, \qquad X \text{ finite.}$$

Observe also that by fixing X we obtain a monad

$$X \,\square\, - \qquad \text{in } FM(\mathbf{Set}).$$

Namely: the free-algebra monad of the above derived signature. And every function $f\colon X \longrightarrow Y$ defines a canonical morphism $f \,\square\, -\colon (X \,\square\, -) \longrightarrow (Y \,\square\, -)$. Thus, $X \,\square\, -$ is an uncurried version of a functor

$$\square\colon \mathbf{Set} \longrightarrow FM(\mathbf{Set}).$$

It is easy see that $\square$ is finitary (in both variables) and thus is a special case of the following

9.4. Definition. (1) (See [U].) *By a **base**, or **parametrized finitary monad**, on $\mathscr{A}$ is meant a finitary functor from $\mathscr{A}$ to $FM(\mathscr{A})$. We use the uncurried notations of $\square\colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$, a functor of two variables defining a monad in the second variable.*

(2) (See [AMV5].) *A **base algebra** is an object A together with a morphism $\alpha\colon A \,\square\, A \longrightarrow A$ forming an Eilenberg-Moore algebra of the monad $A \,\square\, -$.*

More detailed, a base consists of a functor

$$X \,\square\, A \qquad (X, A \in \mathscr{A})$$

of two variables together with natural transformations

$$u_A^X\colon A \longrightarrow X \,\square\, A \qquad \text{(monad unit)}$$

and

$$m_A^X\colon X \,\square\, (X \,\square\, A) \longrightarrow X \,\square\, A \qquad \text{(monad multiplication)}$$

satisfying the usual axioms of a monad $(X \,\square\, -, u^X, m^X)$, and coherence conditions for the variable object X.

9.5. Example. Binary algebras in **Set** lead to three bases (corresponding to the three ways of deciding the iterativity $\mathrm{it}(\sigma)$: two, one, or none).

(a) The base corresponding to the classical concept of E. Nelson ($\mathrm{it}(\sigma) = 2$, i.e., both variables can be iterated) uses the derived signature of constant symbols indexed by $X \times X$. The free algebra on A is

$$X \,\square\, A = (X \times X) + A.$$

The monadic structure is given by $u_A^X =$ right injection and $m_A^X = \nabla_{X\times X} + A\colon (X \times X) + (X \times X) + A \longrightarrow (X \times X) + A$.

(b) The base corresponding to iterating x_1 but not x_2 ($\mathrm{it}(\sigma) = 1$) uses the derived signature of unary operations $\sigma(x, -)$ indexed by $x \in X$. The free algebra on A is

$$X \,\square\, A = X^* \times A$$

where $(X^*, 1 \xrightarrow{u} X^*, X^* \times X^* \xrightarrow{m} X^*)$ is a free monoid. The monadic structure is given by the neutral element:

$$u_A^X \equiv A = 1 \times A \xrightarrow{u \times A} X^* \times A$$

and by the concatenation $c\colon X \times X^* \longrightarrow X^*$:

$$m_A^X\colon X^* \times X^* \times A \xrightarrow{c \times A} X^* \times A.$$

(c) The base corresponding to iterating neither x_1 nor x_2 ($\mathrm{it}(\sigma) = 0$) is

$$X \mathbin{\Box} A = \hat{A}, \text{ free binary algebra on } A$$

(independent of X).

9.6. Definition. (See [AMV5].) *A base algebra* $\alpha\colon A \mathbin{\Box} A \longrightarrow A$ *is called* ***iterative*** *provided that for every* ***finitary equation morphism***

$$e\colon X \longrightarrow X \mathbin{\Box} A \qquad (X \textit{ finitely presentable})$$

there exists a unique ***solution****, i.e., a morphism* $e^\dagger\colon X \longrightarrow A$ *such that the square*

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \hat{\sigma}} \\ X \mathbin{\Box} A & \xrightarrow[e^\dagger \Box A]{} & A \mathbin{\Box} A \end{array}$$

commutes.

9.7. Example. For the base $X \mathbin{\Box} A = (X \times X) + A$ this is precisely the concept of E. Nelson. The base $X \mathbin{\Box} A = X^* \times A$ yields the iterative algebras of Example 9.2. And for the last base $X \mathbin{\Box} A = \hat{A}$ every algebra is iterative.

9.8. Remark. The theory of free iterative algebras and the corresponding free iterative monad, as presented in Sections 6 and 7 above, extend fully to base algebras (but the proofs are technically more involved). In particular:

(a) free iterative base algebras exist,

(b) they can be constructed coalgebraically (as filtered colimits of categories of equations),

(c) the corresponding monad R on $\mathscr{A}$, called the rational monad of the base $\Box$, is iterative in the appropriate sense, and

(d) the rational monad can be characterized by a universal property.

See [AMV5].

10. Coalgebra in Classes

10.1. Remark. All the basic examples of systems mentioned above are coalgebras for endofunctors of **Set**, the category of small sets. Here we extend this category to the category **Class** of classes, and we show how life simplifies with this extension. Our concern is not with questions of foundations of set theory: we work in the "classical" setting with the Axiom of Choice and with a specified universe **Set** of small sets. The Axiom of Choice (assumed for all, not necessarily small, sets) yields a cardinal of the collection of all small sets which we denote by

$$\aleph_\infty.$$

This cardinal is strongly inaccessible, i.e., an uncountable cardinal such that $k < \aleph_\infty$ implies $2^k < \aleph_\infty$. Thus, our foundations are the common classical ones: ZFC with a choice of a strongly inaccessible cardinal $\aleph_\infty$. Observe that the category **Set** is equivalent to the category of all sets of cardinality less than $\aleph_\infty$ (simply because $\aleph_\infty$ is the first large cardinal). This allows us to work with "small" as meaning: smaller than $\aleph_\infty$. Thus, we take **Set** to be the category of all sets of cardinality less than $\aleph_\infty$.

A *class* is a property of sets, i.e., a subset of the (large) set **Set**. Every class is either of cardinality $\aleph_\infty$, or it lies in **Set**. Thus, we take as

Class

the category of all sets of cardinality at most $\aleph_\infty$, equivalent to the usual category of classes.

10.2. Definition. (See [AM].) *An endofunctor H of* **Class** *is called* ***set-based*** *provided that for every class X and every element $b \in HX$ there exists a small subset $m\colon Y \hookrightarrow X$ such that b lies in the image of $Hm\colon HY \hookrightarrow HX$.*

Equivalently: H is accessible (see 3.15) for $\lambda = \aleph_\infty$.

10.3. Theorem. (See [AMV1].) *Every endofunctor of* **Class** *is set-based.*

10.4. Corollary. ([See [AM].) *Every endofunctor of* **Class** *has an initial algebra and a terminal coalgebra.*

In the terminology of P. Freyd [F], this says that **Class** is algebraically complete and cocomplete.

10.5. Corollary. *Every endofunctor of* **Set** *has an extension to an endofunctor of* **Class***, unique up-to a natural isomorphism.*

10.6. Example. The power-set functor $\mathscr{P}$ has the extension $\mathscr{P}'\colon \textbf{Class} \longrightarrow \textbf{Class}$ assigning to every class X the class $\mathscr{P}'X$ of all small subsets of X.

An initial algebra of $\mathscr{P}'$ can be described as the class I of all small sets with the algebraic structure $\mathscr{P}'I \longrightarrow I$ given by assigning to every subset $X \subseteq I$ the same X, as an element of I.

A terminal coalgebra was described in [RT] as the coalgebra of all nonordered trees modulo bisimilarity. In the non-wellfounded set theory the terminal coalgebra is the coalgebra of all small non-wellfounded sets, see [Ac] or [BM].

10.7. Corollary. *Every endofunctor of* **Class** *is iteratable* (*see* 8.3).

In fact, the completely iterative monad generated by any endofunctor of **Class** can be described as a quotient of the tree-monad T_Σ for some signature Σ, as proved in [AMV1].

10.8. Theorem. (See [A2].) *Every endofunctor of* **Class** *satisfies Birkhoff's Covariety Theorem: a collection of coalgebras is presentable by coequations* (*see* 4.18) *iff it is closed under coproducts, subcoalgebras, and quotients.*

10.9. Remark. Surprisingly, Birkhoff's Variety Theorem does not hold in **Class** without limitations: the assumption needed is that the cardinal $\aleph_\infty$ is not measurable, see [AT2].

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